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REAL-TIME CALCULATION OF A CAPUTO FRACTIONAL DERIVATIVE FROM NOISY DATA. THE CASE OF CONTINUOUS MEASUREMENTS

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We consider the problem of finding the derivative of a function, which is a classical problem of mathematical analysis. The values of the function are measured continuously over a finite time interval with some error. We propose an algorithm for the approximate calculation of a Caputo fractional derivative from the measurement values based on the methods of feedback control theory. First, the problem of calculating the fractional derivative is replaced by an inverse problem for a control system. Then the method of dynamic inversion is applied to the inverse problem, which allows us to construct a real-time solution algorithm stable under information noises and computational errors. The algorithm is based on N. N. Krasovskii's extremal aiming method, which is widely known in the theory of guaranteed control, and on a local modification of A. N. Tikhonov's classical regularization method with a smoothing functional. The order of convergence of the proposed algorithm is obtained, and a numerical example illustrating the application of the developed technique for calculating fractional Caputo derivatives of specific functions in real time is considered.

Keywords: Caputo fractional derivative, reconstruction, incomplete information, error estimate.

MSC: 34A08, 49N45, 65D25, 93C40

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