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2020 URAL WORKSHOP ON GROUP THEORY AND COMBINATORICS

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A review of the main events of the 2020 Ural Workshop on Group Theory and Combinatorics, held online on August 24–30, 2020, is presented, and a list of open problems with comments is given. Open problems were formulated by the participants at the Open Problems Session held at the end of the workshop.

Keywords: almost simple group, axial algebra of Jordan type, Cayley graph, chromatic number of a graph, conjugacy class sizes of a finite group, distance-regular graph, finite group, group with non-simple socle, Gruenberg–Kegel graph (prime garph), intersection of nilpotent subgroups, Moore graph, ordinary depth of a subgroup, partitions of a set, spectrum of a finite group, strongly regular graph, totally k-closed group, weak second maximal subgroup.

Н. В. Маслова. Международная конференция “2020 Ural Workshop on Group Theory and Combinatorics”. В статье представлен обзор основных событий Международной конференции “2020 Ural Workshop on Group Theory and Combinatorics”, которая прошла в онлайн формате 24–30 августа 2020 г. Также в статье представлен список открытых проблем, сформулированных участниками на прошедшем в конце конференции Часе открытых проблем, и комментарии к этим проблемам.

Ключевые слова: почти простая группа, аксиальная алгебра йорданова типа, граф Кэли, хроматическое число графа, размеры классов сопряженности конечной группы, дистанционно регулярный граф, конечная группа, группа с непростым цоколем, граф Грюнберга — Кегеля (граф простых чисел), пересечение нильпотентных подгрупп, граф Мура, обыкновенная глубина подгруппы, разбиения множества, спектр конечной группы, сильно регулярный граф, вполне 2-замкнутая группа, слабо вторая максимальная подгруппа.

MSC: 17A99, 17C27, 20B15, 05C12, 05C15, 05C25, 05C50, 05C99, 05E18, 05E30, 20B05, 20B25, 20C15, 20D05, 20D06, 20D08, 20D15, 20D20, 20D30, 20D60, 20D99, 20E15, 20E32, 20E45, 20F29, 51E20

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The 2020 Ural Workshop on Group Theory and Combinatorics was held online on August 24–30, 2020. The workshop was organized by the Institute of Natural Sciences and Mathematics of the Ural Federal University named after the first President of Russia B.N. Yeltsin, N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, and the Regional Scientific and Educational Mathematical Center “Ural Mathematical Center”.

The main goal of the event was to bring together the leading and young researchers from all over the world working in group theory, combinatorics, and their applications. The workshop topics covered modern aspects of group theory (including questions of actions of groups on combinatorial objects), graph theory, theory of non-associative algebras, algebraic combinatorics, some combinatorial aspects of topology and optimization theory, and related issues. The program of the workshop included 22 fifty-minute talks by keynote speakers and 49 twenty-minute contributed talks.

Keynote speakers (in alphabetical order) and their talks:

Rosemary A. Bailey (The University of St Andrews, St Andrews, UK), *‘Latin cubes’*

Peter J. Cameron (The University of St Andrews, St Andrews, UK), *‘From de Bruijn graphs to automorphisms of the shift’*

Ilya B. Gorshkov (Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia), *‘On Thompson’s conjecture for finite simple* groups’

Tatsuro Ito (Kanazawa University, Kanazawa, Japan; Anhui University, Hefei, China), *'The Weisfeiler–Leman stabilization revisited from the viewpoint of Terwilliger algebras'*

Alexander A. (Sasha) Ivanov (Imperial College London, London, UK), *'Densely embedded subgraphs in locally projective graphs'*

Vladislav V. Kabanov (N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia), *'On strongly Deza graphs'*

Mikhail Yu. Khachay (N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia; Ural Federal University, Yekaterinburg, Russia), *'Efficient approximation of vehicle routing problems in metrics of a fixed doubling dimension'*

Elena V. Konstantinova (Novosibirsk State University, Novosibirsk, Russia; Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia), *'Dual Seidel switchings and integral graphs'*

Denis S. Krotov (Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia), *'On the parameters of unrestricted completely regular codes'*

Long Miao (Yangzhou University, Yangzhou, China), *'On second maximal subgroups of finite groups'*

Christopher Parker (University of Birmingham, Birmingham, UK), *'Subgroups like minimal parabolic subgroups'*

Ilia Ponomarenko (Petersburg Department of V.A. Steklov Institute of Mathematics, Saint Petersburg, Russia), *'On the Weisfeiler–Leman dimension of Paley graphs'*

Cheryl E. Praeger (The University of Western Australia, Perth, Australia), *'Totally* 2*-closed groups'*

Danila O. Revin (Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia; N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia), *'Reduction theorems for relatively maximal subgroups'*

Wujie Shi (Chongqing University of Arts and Sciences, Chongqing, China; Suzhou University, Suzhou, China), *'On the widths of finite groups'*

Sergey Shpectorov (University of Birmingham, Birmingham, UK), *'Generalised Sakuma theorem'*

Arseny Shur (Ural Federal University, Yekaterinburg, Russia), *'Words separation and positive identities in symmetric groups'*

Alexey Staroletov (Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia), *'On axial algebras of Jordan type'*

Vladimir I. Trofimov (N.N. Krasovskii Institute of Mathematics and Mechanics UB RAS, Yekaterinburg, Russia; Ural Federal University, Yekaterinburg, Russia), *'Cayley graphs among vertex-symmetric graphs'*

Mikhail V. Volkov (Ural Federal University, Yekaterinburg, Russia), *'Computational complexity of synchronization under regular constraints'*

Yaokun Wu (Shanghai Jiao Tong University, Shanghai, China), *'Digraph homomorphisms with the path-lifting property'*

Alexandre Zalesskii (University of East Anglia, Norwich, UK), *'Two projects in the representation theory of finite simple groups'*

The Open Problems Session was held on the last day of the workshop. The participants posed a number of open problems in their areas of research. Here we list the problems and some comments on recent progress in their investigation. The records of all discussions are available on the workshop website [45] for registered participants and on the website of the Ural Seminar on Group Theory and Combinatorics [44] after registration. The problems below are ordered so as to avoid repetition of definitions and notation.

1. Let G be a group and m a positive integer. The diagonal graph $GD(G,m)$ is the Cayley graph of G^m with connection set consisting of the non-identity elements of the subgroups G_i, for $i = 1, \dots, m+1$, where

(1) for $i = 1, \dots, m$, $G_i = \{(g_1, ..., g_m) \mid g_j = 1 \text{ for } j \neq i\}$;

(2) G_{m+1} is the diagonal subgroup $\{(g, g, ..., g) \mid g \in G\}$.

For $m = 2$, $GD(G, 2)$ is the Latin square graph associated with the Cayley table of G. Assume that $m \geq 3$. The clique number of the graph $GD(G,m)$ is $|G|$; the cosets of each subgroup G_i are cliques.

We ask, what is the chromatic number of $GD(G,m)$? We know that the answer is $|G|$ if m is odd, or if $|G|$ is odd, or if the Sylow 2-subgroups of G are non-cyclic. (The last fact uses the truth of the Hall–Paige conjecture, a consequence of the Classification of Finite Simple Groups.) What happens in the remaining case? For example, if $G \cong C_2$ and m is even, then the chromatic number of $GD(G,m)$ is $4 = |G| + 2$.

See for references [3; 4].

Rosemary A. Bailey and Peter J. Cameron

2. Let m and r be integers with $m \geq 2$ and $r \geq 1$. When is there a set $\mathfrak{Q}$ of $m+r$ partitions of a set Ω with the property that any m of these partitions are the minimal non-trivial partitions in a Cartesian lattice of dimension m?

If $m = 2$ and $r = 1$, then these are precisely Latin squares.

If $m = 2$ and $r > 1$, then these are precisely sets of r mutually orthogonal Latin squares.

If $m \geq 3$ and $r = 1$, then there is a unique group G such that $\Omega = G^m$ and the partitions give the right cosets of the m coordinate subgroups and the diagonal subgroup $d(G) = \{(g, g, ..., g) \mid g \in G\}$. (When $m = 3$ these are precisely regular Latin cubes of sort (LC2).)

An example with $m = 4$ and $r = 2$. Let $p \geq 5$ be a prime. Put

$$G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \cong C_p^4.$$

Then the coset partitions of $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle d \rangle$, $\langle abcd \rangle$, and $\langle ab^2c^3d^4 \rangle$ have the required property.

More generally, let V be an m-dimensional vector space over a finite field, and let $v_1, \dots, v_{m+r}$ be vectors any m of which are linearly independent. Then the coset partitions of the subspaces $\langle v_1 \rangle, \dots, \langle v_{m+r} \rangle$ have the required property. It is known that, in the case when $m = 3$, the maximal cardinality of such a set of partitions over the field of order q is $q + 1$ if q is odd, or $q + 2$ if q is even. (Here finite geometers can recognise connections with arcs and MDS codes.)

Can we characterize all such combinatorial objects?

Rosemary A. Bailey and Peter J. Cameron

The following notation will be used in the next three problems. Let G be a finite group. For $g \in G$, denote by g^G the conjugacy class of G containing g and by $|g^G|$ the size of the conjugacy class g^G. Let $N(G) = \{|g^G| \mid g \in G\}$.

3. A finite group G with trivial center is *recognizable by the set of conjugacy class sizes among finite groups with trivial center* if the equality $N(G) = N(H)$, where H is a finite group with trivial center, implies the isomorphism $H \cong G$.

In 1987 J. Thompson conjectured that if G is a finite nonabelian simple group, H is a finite group with trivial center, and $N(G) = N(H)$, then $H \cong G$, see Problem 12.38 in the Kourovka Notebook [35]. This conjecture was studied for different groups in a series of papers and was finally proved in [19].

Let G be a finite nonabelian simple group. Is it true that, for each integer $n \geq 1$, the direct product G^n of n copies of G is recognizeble by the set of conjugacy class sizes among finite groups with trivial center? In particular, is it true that the group $J_4 \times J_4$ is recognizeble by the set of conjugacy class sizes among finite groups with trivial center?

Ilya B. Gorshkov

4. (*A question initiated by Peter J. Cameron.*) Let G be a finite group and $\alpha = \max N(G)$. A.A. Buturlakin[1] proved that there is a function f such that $|G|/|Z(G)| \leq f(\alpha)$. Does there exist a constant β such that $|G|/|Z(G)| \leq \alpha^\beta$?

Recent progress. I.B. Gorshkov has informed the author of this survey paper that there are some known results about this problem, for example, see [42].

Ilya B. Gorshkov

5. Let G be a finite group and p be a prime. Denote by $|G||_p$ the number p^n such that $N(G)$ contains a multiple of p^n but does not contain a multiple of p^{n+1}. Let G contains a p-element g such that $|g^G| = |G||_p$. Is it true that G has a normal p-complement? This is true if for every $\alpha \in N(G)$ we have $\alpha_p \in \{1, |G||_p\}$, where α_p is the p-part of α (see [20]).

Ilya B. Gorshkov

6. Let A be a simple primitive finitely generated axial algebra of Jordan type 1/2. Is A a unital algebra? When is the Miyamoto group of A simple?

See [23; 26] for some references.

Ilya B. Gorshkov

7. (*A question by László Héthelyi.*) Let G be a finite group, and let H be a subgroup of G. Consider a simple graph Γ whose vertex set is $\mathrm{Irr}(H)$ and two vertices ϕ and ψ are adjacent if there exists $\chi \in \mathrm{Irr}(G)$ such that $(\chi_H, \phi) \neq 0$ and $(\chi_H, \psi) \neq 0$. Let $d(\phi, \psi)$ be the standard distance between ϕ and ψ in Γ (note that we have $d(\phi, \psi) = -\infty$ if ϕ and ψ belong to distinct connected components of Γ). The ordinary depth $d(H, G)$ (see [10]) is the smallest positive integer which can be determined from the following upper bounds:

(1) For $m \geq 1$ the inequality $d(H, G) \leq 2m + 1$ holds if and only if $d(\phi_i, \phi_j) \leq m$ for every $\phi_i, \phi_j \in \mathrm{Irr}(H)$.

(2) For $m \geq 2$ the inequality $d(H, G) \leq 2m$ holds if and only if

$$\max_{\alpha \in \mathrm{Irr}(H)} d(\alpha, \chi_H) \leq m - 1 \text{ for every } \chi \in \mathrm{Irr}(G).$$

(3) $d(H, G) \leq 2$ if and only if H is normal in G.

(4) $d(H, G) = 1$ if and only if $G = HC_G(x)$ for all $x \in H$.

[1]Communication. This result by A.A. Buturlakin was announced in the talk by I. B. Gorshkov at the 2020 Ural Workshop on Group Theory and Combinatorics.

Recall that $\mathrm{Core}_G(H)$ is the largest normal subgroup of G contained in H. It is known [10, Theorem 6.9] that if $\mathrm{Core}_G(H) = \cap_{i=1}^m H^{x_i}$ for $x_i \in G$, then $d(H,G) \le 2m$. If additionally $\mathrm{Core}_G(H) \le Z(G)$, then $d(H,G) \le 2m-1$. Therefore, if H has a disjoint conjugate, then $d(H,G) \le 3$. The converse is not true, see, for example, A_5 in A_7.

If $d(H,G)=3$, then is $\mathrm{Core}_G(H)$ the intersection of at most 3 conjugates of H?

See for references also [7; 8; 15; 16; 27; 28; 30; 33; 34].

Erzsébet Horváth

8. Let Δ be the unique distance-regular graph with intersection array $\{32,27;1,12\}$ (or equivalently, the unique strongly regular graph with parameters $(105,32,4,12)$). Parameter feasibility conditions (see, for example, [9, Chapter 4.1.D]) imply that a distance-regular antipodal r-cover of Δ, where r is a positive integer, has diameter 4 and intersection array $\{32,27,\frac{12(r-1)}{r}, 1;1,\frac{12}{r},27,32\}$ for $r \in \{2,3,4,6\}$.

Now let Γ be a distance-regular graph with intersection array

$$\{32,27,\frac{12(r-1)}{r},1;1,\frac{12}{r},27,32\} \text{ for } r \in \{2,3,4,6\}.$$

It is known that if $r \in \{2,4\}$, then such a graph Γ does not exist, and if $r=3$, then such a graph Γ exists and is unique (see [43]). Does such a graph Γ exist in the case $r=6$?

Alexander A. Makhnev

Recent progress. A.A. Makhnev has informed the author of this survey paper that recently D.V. Paduchikh and A.A. Makhnev have proved that if $r=6$, then such a graph Γ does not exist.

9. Let $n=2^e \ge 8$ be an integer. Then the bilinear forms graph $\mathrm{Bil}_2(3\times e)$ is a distance-regular graph with intersection array $\{7(n-1),6(n-2),4(n-4);1,6,28\}$ (see [9]). These graphs are uniquely determined by their parameters up to isomorphism (see [17; 40]).

The intersection array $\{7(n-1),6(n-2),4(n-4);1,6,28\}$ is feasible (see [9, Chapter 4.1.D]) for all integers $n \ge 6$. By [40, Corollary 1.3(d)], if there exists a distance-regular graph with intersection array $\{7(n-1),6(n-2),4(n-4);1,6,28\}$ which is not a bilinear forms graph, then $n \le 133$. This bound was improved to $n \le 70$ for arbitrary distance-regular graphs with intersection array $\{7(n-1),6(n-2),4(n-4);1,6,28\}$ and to $n \le 42$ for geometric graphs (see [18]). The case $n=6$ was ruled out in [32] and independently in [17]; the case $n=7$ was ruled out in [6]. Does such a graph exist in the case $n=9$?

Alexander A. Makhnev

10. (*Well-known question of existence of a Moore graph of degree* 57.) If a regular graph of degree k and diameter d has v vertices, then

$$v \le 1+k+k(k-1)+\ldots+k(k-1)^{d-1}.$$

Regular graphs of degree k and diameter d with $v = 1+k+k(k-1)+\ldots+k(k-1)^{d-1}$ are called Moore graphs.

If Γ is a Moore graph of degree k and diameter d, then either $k=2$, or $d=2$, $v=k^2+1$, Γ is strongly regular with parameters $(k^2+1,k,0,1)$, and one of the following statements holds: $k=2$ and Γ is the pentagon, $k=3$ and Γ is the Petersen graph, $k=7$ and Γ is the Hoffman-Singleton graph, or $k=57$ (for details see [5; 14; 29] and the survey paper [13]).

Does a Moore graph of degree 57 exist?

A. Jurišić and J. Vidali [31] proved that a Moore graph of degree 57 exists if and only if a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$ exists.

Does a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$ exist?

Alexander A. Makhnev

Recent progress. A.A. Makhnev has put the Russian preprint [37] to ArXiv; there he proved that a distance-regular graph with intersection array $\{55, 54, 2; 1, 1, 54\}$ does not exist, which implies that a Moore graph of degree 57 does not exist. We are looking forward to the English version of this manuscript.

The following notation will be used in the next three problems. Let G be a finite group. The *spectrum* $\omega(G)$ is the set of all element orders of G.

11. A finite group G is *recognizable by spectrum* if for each finite group H, $\omega(G) = \omega(H)$ if and only if $G \cong H$. It is known that many finite nonabelian simple groups are recognizable by spectrum, see surveys in the papers [24; 25].

There are examples of finite groups with non-simple socle which are recognizable by spectrum. Namely, the groups $Sz(2^7) \times Sz(2^7)$ (see [38]) and $J_4 \times J_4$ (see [22]) are recognizable by spectrum.

If the direct product of k copies of a finite group G is recognizable by spectrum, then, for each $i \leq k$, the direct product of i copies of G is recognizable by spectrum. Moreover, if G is a finite group, then there exists $r = r(G)$ such that, for each $i \geq r$, the direct product of i copies of G has the same spectrum as a finite abelian group; therefore, by [39], the direct product of $i \geq r(G)$ copies of G is not recognizable by spectrum.

Let G be a finite group which is recognizable by spectrum. What is the largest number $k = k(G)$ such that the direct product of k copies of the group G is still recognizable by spectrum?

Is it true that for each integer $k \geq 1$ there exists a finite simple group $G = G(k)$ such that the direct product of k copies of G is recognizable by spectrum?

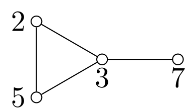
Natalia V. Maslova

Recent progress. Recently, I.B. Gorshkov has proved that if $m > 5$, then the group $PSL_{2^m}(2) \times PSL_{2^m}(2) \times PSL_{2^m}(2)$ is recognizable by spectrum (see [21]).

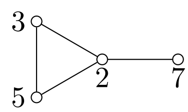
12. For which values of q the group $E_8(q) \times E_8(q)$ is recognizable by spectrum?

Ilya B. Gorshkov

13. Let G be a finite group. The *prime spectrum* $\pi(G)$ is the set of all prime divisors of $|G|$. A simple graph $\Gamma(G)$ with vertex set $\pi(G)$ in which two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$ is called the *Gruenberg–Kegel graph* or the *prime graph* of G. It is easy to see that for finite groups G and H, if $G \cong H$, then $\omega(G) = \omega(H)$; if $\omega(G) = \omega(H)$, then $\Gamma(G) = \Gamma(H)$; if $\Gamma(G) = \Gamma(H)$, then $\Gamma(G)$ and $\Gamma(H)$ are isomorphic as abstract graphs and $\pi(G) = \pi(H)$. The converse does not hold in each case: $S_5 \not\cong S_6$ but $\omega(S_5) = \omega(S_6)$; $\omega(A_5) \neq \omega(A_6)$ but $\Gamma(A_5) = \Gamma(A_6)$; $\Gamma(A_{10}) \neq \Gamma(\mathrm{Aut}(J_2))$ but $\Gamma(A_{10})$ and $\Gamma(\mathrm{Aut}(J_2))$ are isomorphic as abstract graphs and $\pi(A_{10}) = \pi(\mathrm{Aut}(J_2))$, see the picture below:



$\Gamma(A_{10})$,



$\Gamma(\mathrm{Aut}(J_2))$.

A finite group G is *recognizable by the isomorphism type of its Gruenberg–Kegel graph* if, for each finite group H, the graphs $\Gamma(G)$ and $\Gamma(H)$ are isomorphic as abstract graphs if and only if $G \cong H$. The simple sporadic group J_4 is the unique finite group whose Gruenberg–Kegel graph has exactly six connected components (see [47, Theorem B]); therefore, J_4 is recognizable by the isomorphism type of its Gruenberg–Kegel graph.

What are finite groups which are recognizable by the isomorphism types of their Gruenberg–Kegel graphs?

Natalia V. Maslova

Recent progress. It follows from [11, Theorem 1.3] that if a group G is recognizable by the isomorphism type of its Gruenberg–Kegel graph, then G is almost simple. By [11, Theorem 1.5], the groups $E_8(2)$ and ${}^2G_2(27)$ are recognizable by isomorphism types of their Gruenberg–Kegel graphs. Recently the author of this paper with the participation of M.R. Zinovieva has proved that the groups $E_8(q)$, where $q \in \{3, 4, 5, 7, 8, 9, 17\}$, are recognizable by the isomorphism types of their Gruenberg–Kegel graphs.[2] Thus, the following questions naturally arise:

Are there other finite simple groups which are recognizable by the isomorphism types of their Gruenberg–Kegel graphs?

Is there an almost simple but not simple group which is recognizable by the isomorphism type of its Gruenberg–Kegel graph?

14. Let G be a finite group. A subgroup H of G is a *second maximal subgroup* of G if there is a maximal subgroup M of G such that H is maximal in M. If H is a second maximal subgroup of G, then define

$$\mathrm{Max}(G, H) = \{M \mid M \text{ is maximal in } G \text{ and } H \leq M\}.$$

A subgroup H is a *strong second maximal subgroup* in G if H is maximal in each $M \in \mathrm{Max}(G, H)$ and is a *weak second maximal subgroup* otherwise. Each finite group has at least one strong second maximal subgroup.[3] Let M be a maximal subgroup of G. What are pairs (G, M) such that all second maximal subgroups H of G with $H \leq M$ are weak second maximal subgroups of G?

Long Miao

15. Let G be a permutation group on a set Ω. The *k-closure* $G^{(k),\Omega}$ of G is the set of all $g \in \mathrm{Sym}(\Omega)$ (permutations of Ω) such that g leaves invariant each G-orbit in the induced G-action on ordered k-tuples from Ω. The k-closure $G^{(k),\Omega}$ is a subgroup of $\mathrm{Sym}(\Omega)$ containing G, and a permutation group G is said to be *k-closed* if $G^{(k),\Omega} = G$. Different faithful permutation representations of the same group G may have quite different k-closures.

For a positive integer k, a group G is said to be *totally k-closed* if $G^{(k),\Omega} = G$ whenever G is faithfully represented as a permutation group on Ω. If $G \leq Sym(\Omega)$ is a permutation group and $k \geq 2$, then by [46, Theorem 5.8],

$$G \leq G^{(k),\Omega} \leq G^{(k-1),\Omega}.$$

Thus, if G is totally $(k-1)$-closed, then it is totally k-closed.

Does there exist a finite totally 3-closed group which is not totally 2-closed?

[2]This work was supported by the Russian Science Foundation (project no. 19-71-10067).

[3]This result was announced in Long Miao's talk at the 2020 Ural Workshop on Group Theory and Combinatorics.

For which values of k does there exist a finite totally k-closed group which is not totally $(k-1)$-closed?

For references see also [1; 2; 36; 41].

Cheryl E. Praeger

Recent progress. The talk by Cheryl E. Praeger at the 2020 Ural Workshop on Group Theory and Combinatorics and discussions during the Open Problems Session gave rise to her cooperation with Dmitry Churikov, who is a doctoral student from Novosibirsk. Recently D. Churikov and C.E. Praeger [12] have proved that if $k \geq 2$ is an integer, then for each prime p, there are infinitely many finite abelian p-groups which are totally k-closed but not totally $(k-1)$-closed.

16. Let G be a finite group, and let A and B be subgroups of G. Denote by $M_G(A,B)$ the set of minimal by inclusion intersections of the form $A \cap B^g$ for $g \in G$, and by $m_G(A,B)$ the set of minimal by order intersections of the form $A \cap B^g$ for $g \in G$. It is clear that $m_G(A,B) \subseteq M_G(A,B)$. Define the subgroups $\mathrm{Min}_G(A,B) = \langle M_G(A,B) \rangle$ and $\mathrm{min}_G(A,B) = \langle m_G(A,B) \rangle$ of G. It is clear that $\mathrm{min}_G(A,B) \leq \mathrm{Min}_G(A,B)$ and $\mathrm{min}_G(A,B) \neq 1$ if and only if $\mathrm{Min}_G(A,B) \neq 1$. If A and B are abelian subgroups of G, then $\mathrm{Min}_G(A,B) \leq F(G)$, where $F(G)$ is the Fitting subgroup of G (see [48]). Is it true that $\mathrm{min}_G(A,B) \leq F(G)$ if A is an abelian subgroup of odd order and B is a nilpotent subgroup of G?

Viktor I. Zenkov

Recent progress. V.I. Zenkov has informed the author of this survey paper that in some partial cases the problem is solved in the positive.

The 2020 Ural Workshop on Group Theory and Combinatorics is continued by the Ural Seminar on Group Theory and Combinatorics, which will be held every other Tuesday with possible changes and exceptions. The list of talks of the seminar can be found on its website [44].

We are looking forward to meeting you at the Ural Seminar on Group Theory and Combinatorics!

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