Trudy Instituta Matematiki i Mekhaniki UrO RAN

Vol. 27 No. 1 2021

УДК 512.542

CONTINUATION OF THE THEORY OF $E_{\mathfrak{F}}$-GROUPS[1]

I. L. Sokhor

We describe the structure of finite groups with $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups when $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups. We prove that groups with absolutely $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups are soluble when $\mathfrak{F}$ is a subgroup-closed saturated formation containing all nilpotent groups.

Keywords: finite group, primary cyclic subgroup, subnormal subgroup, abnormal subgroup, derived subgroup.

И. Л. Сохор. Развитие теории конечных $E_{\mathfrak{F}}$-групп.

Описана структура конечных групп с $\mathfrak{F}$-субнормальными или самонормализуемыми примарными циклическими подгруппами в случае, когда $\mathfrak{F}$ — наследственная насыщенная сверхрадикальная формация, содержащая все нильпотентные группы. Доказано, что группы с абсолютно $\mathfrak{F}$-субнормальными или самонормализуемыми примарными циклическими подгруппамии разрешимы, если $\mathfrak{F}$ — наследственная насыщенная формация, содержащая все нильпотентные группы.

Ключевые слова: конечная группа, примарная циклическая подгруппа, субнормальная подгруппа, абнормальная подгруппа, коммутант.

MSC: 20D10; 20D35

DOI: 10.21538/0134-4889-2021-27-1-268-272

1. Introduction

All groups in this paper are finite. We use the standard notation and terminology of [1].

Let $\mathfrak{F}$ be a formation and let G be a group.

A subgroup H of G is $\mathfrak{F}$-subnormal in G if $G = H$ or there is a chain of subgroups

$$H = H_0 \lessdot H_1 \lessdot \ldots \lessdot H_n = G$$

such that $H_i/(H_{i-1})_{H_i} \in \mathfrak{F}$ for all i (or, equivalently, $H_i^{\mathfrak{F}} \le H_{i-1}$). Here we write $A \lessdot B$ if A is a maximal subgroup of a group B and denote by $A_B = \bigcap_{b\in B} A^b$ the core of A in B.

A subgroup H of G is $\mathfrak{F}$-abnormal in G if $L/K_L \notin \mathfrak{F}$ for all K and L such that $H \le K \lessdot L \le G$.

A group G is said to be an $E_{\mathfrak{F}}$-group if $G \notin \mathfrak{F}$ and its every non-trivial subgroup is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal in G. The structure of $E_{\mathfrak{F}}$-groups for various formations $\mathfrak{F}$ has been studied by many authors, see the review paper by A. N. Skiba [2].

It is evident that in a group any proper subgroup cannot be $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal at the same time; i. e. these properties are alternative. If $\mathfrak{F}$ is a subgroup-closed formation containing all nilpotent subgroups, then every $\mathfrak{F}$-abnormal subgroup is self-normalizing; i. e. it coincides with its normalizer. But the properties of being self-normalizing and $\mathfrak{F}$-subnormal are not alternative. For example, every non-normal subgroup of prime index in a soluble group is self-normalizing and $\mathfrak{U}$-subnormal. Here $\mathfrak{U}$ is the formation of all supersoluble groups.

Groups with $\mathfrak{F}$-subnormal or self-normalizing subgroups were studied in [3; 4]. In particular, V. S. Monakhov [3] showed that the class of groups with $\mathfrak{U}$-subnormal or self-normalizing primary subgroups is much wider than the class of $E_{\mathfrak{U}}$-groups.

[1]This paper is based on the results of the 2020 Ural Workshop on Group Theory and Combinatorics.

In this paper, we continue the research on this topic. We describe the structure of groups with $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups when $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups. In addition, we prove that groups with absolutely $\mathfrak{F}$-subnormal or self-normalizing primary cyclic subgroups are soluble when $\mathfrak{F}$ is a subgroup-closed saturated formation containing all nilpotent groups.

2. Preliminaries

Let G be a group. We denote the set of all prime devisors of $|G|$ by $\pi(G)$, $A \rtimes B$ denotes the semidirect product of a normal subgroup A and a subgroup B.

The formations of all abelian and nilpotent groups are denoted by $\mathfrak{A}$ and $\mathfrak{N}$, respectively.

A normal subgroup-closed formation $\mathfrak{F}$ is called superradical if any group $G = AB$ belongs to $\mathfrak{F}$ whenever A and B are $\mathfrak{F}$-subnormal $\mathfrak{F}$-subgroups of G. It is known that formations with Shemetkov property and lattice formations are superradical.

Let $\mathfrak{F}$ be a formation and let G be a group. The intersection of all normal subgroups of G with quotient in $\mathfrak{F}$ is called the $\mathfrak{F}$-residual and is denoted by $G^{\mathfrak{F}}$.

We need the following properties of $\mathfrak{F}$-subnormal and $\mathfrak{F}$-abnormal subgroups.

Lemma 1. *Let $\mathfrak{F}$ be a formation, let H and K be subgroups of a group G, and let N be a normal subgroup of G. The following statements hold.*

(1) *If K is $\mathfrak{F}$-subnormal in H and H is $\mathfrak{F}$-subnormal in G, then K is $\mathfrak{F}$-subnormal in G* [5, 6.1.6 (1)].

(2) *If $\mathfrak{F}$ is a subgroup-closed formation and $G^{\mathfrak{F}} \leq H$, then H is $\mathfrak{F}$-subnormal in G* [5, 6.1.7 (1)].

(3) *If $\mathfrak{F}$ is a subgroup-closed formation, $K \leq H$, H is $\mathfrak{F}$-subnormal in G and $H \in \mathfrak{F}$, then K is $\mathfrak{F}$-subnormal in G.*

Proof. (3) Since $\mathfrak{F}$ is a subgroup-closed formation and $K \leq H$, we have K is $\mathfrak{F}$-subnormal in H. In view of Statement (1), we conclude H is $\mathfrak{F}$-subnormal in G. □

Lemma 2 [3, Lemma 1.4]. *Let $\mathfrak{F}$ be a subgroup-closed formation containing groups of order p for all $p \in \mathbb{P}$, and let A be an $\mathfrak{F}$-abnormal subgroup of a group G. The following statements hold.*

(1) *If $A \leq B \leq G$, then B is $\mathfrak{F}$-abnormal in G and $B = N_G(B)$.*

(2) *If G is soluble, then A is abnormal in G.*

A subgroup H of a group G is abnormal if $x \in \langle H, H^x \rangle$ for any $x \in G$. An abnormal subgroup is self-normalizing.

It is easy to check.

Lemma 3. *Let G be a group. The following statements hold.*

(1) *If P is a Sylow subgroup of G, then $N_G(P)$ is abnormal in G.*

(2) *If A is an abnormal subgroup of G and $A \leq B \leq G$, then B is abnormal in G and $N_G(B) = B$.*

A Carter subgroup is a nilpotent self-normalizing subgroup [1, VI.12]. In soluble groups, Carter subgroups exist and are conjugate. An insoluble group can have no Carter subgroups, but if they exist, they are conjugate [6].

A group G is a minimal non-$\mathfrak{F}$-group if $G \notin \mathfrak{F}$ but every proper subgroup of G belongs to $\mathfrak{F}$. A minimal non-$\mathfrak{N}$-group is also called a Schmidt group; its properties are well known [7].

Lemma 4. *Let $\mathfrak{F}$ be a saturated formation. If every maximal subgroup of a group G is $\mathfrak{F}$-subnormal, then $G \in \mathfrak{F}$.*

Proof. If M is a maximal subgroup of G, then $G/M_G \in \mathfrak{F}$. Hence $G/\bigcap M_G = G/\Phi(G) \in \mathfrak{F}$ and $G \in \mathfrak{F}$. □

3. Groups with $\mathfrak{F}$-subnormal or self-normalizing primary subgroups

Lemma 5. *Let $\mathfrak{F}$ be a subgroup-closed saturated superradical formation containing all nilpotent groups. A soluble group G belongs to $\mathfrak{F}$ if and only if every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal.*

Proof. If $G \in \mathfrak{F}$, then every proper subgroup (including every primary cyclic subgroup) is $\mathfrak{F}$-subnormal in G.

Now, suppose that there are groups such that they do not belong to $\mathfrak{F}$ but all their primary cyclic subgroup are $\mathfrak{F}$-subnormal. Let G be a gruop of least order among them. Then every proper subgroup of G belongs to $\mathfrak{F}$. According to [8, Lemma 3], G is a Schmidt group, and $G = P \rtimes \langle y\rangle$ [7, Theorem 1.1]. In view of [7, Theorem 1.5], either $G^{\mathfrak{F}} \leq \Phi(G)$ or $P \leq G^{\mathfrak{F}}$. If $G^{\mathfrak{F}} \leq \Phi(G)$, then $G \in \mathfrak{F}$ as $\mathfrak{F}$ is a saturated formation, a contradiction. Assume that $P \leq G^{\mathfrak{F}}$. By the choice of G, $\langle y\rangle$ is $\mathfrak{F}$-subnormal in G. Therefore, there is a maximal subgroup M in G that contains $\langle y\rangle$ and $G^{\mathfrak{F}}$, a contradiction. □

Theorem 1. *If $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups and $G \notin \mathfrak{F}$ is a soluble group, then the following statements are equivalent.*

(1) *Every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal or self-normalizing.*

(2) *Every non-abnormal subgroup of G is $\mathfrak{F}$-subnormal and belongs to $\mathfrak{F}$.*

(3) *$G = G' \rtimes \langle x\rangle$, where $\langle x\rangle$ is a Sylow p-subgroup for a prime $p \in \pi(G)$ and a Carter subgroup, $G' = G^{\mathfrak{N}}$, $G' \rtimes \langle x^p\rangle \in \mathfrak{F}$.*

Proof. (1) ⇒ (3): This is true in view of [4, Corollary 3.3.2], but for completeness, we give a direct proof.

Assume that every primary cyclic subgroup of a soluble group $G \notin \mathfrak{F}$ is $\mathfrak{F}$-subnormal or self-normalizing. By Lemma 5, for some $p \in \pi(G)$ there is a cyclic p-subgroup $\langle x\rangle$ that is not $\mathfrak{F}$-subnormal in G. By the choice of G, $\langle x\rangle$ is self-normalizing. Hence $\langle x\rangle$ is a Sylow subgroup and a Carter subgroup of G. In view of [1, IV.2.6], there is a normal Hall p'-subgroup $G_{p'}$ of G, and $G = G_{p'} \rtimes \langle x\rangle$. Clearly, $G^{\mathfrak{N}} \leq G' \leq G_{p'}$. Since $G/G^{\mathfrak{N}}$ is nilpotent and $PG^{\mathfrak{N}}/G^{\mathfrak{N}}$ is a Sylow subgroup of $G/G^{\mathfrak{N}}$, we conclude that $PG^{\mathfrak{N}}$ is normal in G. In view of the Frattini lemma,

$$G = N_G(P)(PG^{\mathfrak{N}}) = PG^{\mathfrak{N}} = G_{p'} \rtimes P.$$

So $G^{\mathfrak{N}} = G' = G_{p'}$ and $G = G' \rtimes \langle x\rangle$.

By [1, VI.12.2], $G' \rtimes \langle x^p\rangle$ has no self-normalizing primary cyclic subgroups. Hence, it follows from the choice of G that every primary cyclic subgroup of $G' \rtimes \langle x^p\rangle$ is $\mathfrak{F}$-subnormal in G, and $G' \rtimes \langle x^p\rangle \in \mathfrak{F}$ according to Lemma 5.

(3) ⇒ (2): Assume that a soluble group $G \notin \mathfrak{F}$ satisfies statement (3). Let H be a non-abnormal subgroup of G. By the choice of G, we have $G' \in \mathfrak{F}$. Since $\mathfrak{A} \subseteq \mathfrak{N} \subseteq \mathfrak{F}$ and $\mathfrak{F}$ is a subgroup-closed formation, it follows that G' is $\mathfrak{F}$-subnormal in G by Lemma 1 (2). If $H \leq G'$, then H is $\mathfrak{F}$-subnormal in G by Lemma 1 (3). Suppose that H is not contained in G'. If $G = G'H$, then H contains a subgroup $\langle x\rangle$ such that $\langle x\rangle$ is a Sylow p-subgroup for a prime $p \in \pi(G)$ and a Carter subgroup. Hence $\langle x\rangle$ is abnormal in G by Lemma 3 (1), and H is abnormal in G by Lemma 3 (2). This contradicts the choice of H. Hence $G'H$ is a proper subgroup of G. According to the choice of G, $G'H \in \mathfrak{F}$ and H is $\mathfrak{F}$-subnormal in G by Lemma 1 (3).

(2) ⇒ (1): The implication is obvious because abnormal subgroups are self-normalizing. □

Corollary 1. *Let $\mathfrak{F}$ be a subgroup-closed saturated superradical formation containing all nilpotent groups. Assume that every primary cyclic subgroup of a soluble group $G \notin \mathfrak{F}$ is $\mathfrak{F}$-subnormal or self-normalizing, K is a Carter subgroup of G and A is a proper subgroup of G. The following statements hold.*

(1) *If $|K|$ divides $|A|$, then A is abnormal.*

(2) *If $|K|$ does not divide $|A|$, then A is $\mathfrak{F}$-subnormal and $A \in \mathfrak{F}$.*

Note that if $\mathfrak{F}$ is a subgroup-closed formation containing all nilpotent subgroups, then in view of Lemma 2 every $\mathfrak{F}$-abnormal subgroup is self-normalizing. Hence Theorem 1 bellows us to describe the structure of an $E_{\mathfrak{F}}$-group when $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups.

Corollary 2. *If $\mathfrak{F}$ is a subgroup-closed saturated superradical formation containing all nilpotent groups and $G \notin \mathfrak{F}$ is a soluble group, then the following statements are equivalent.*

(1) *Every primary cyclic subgroup of G is $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal.*

(2) *G is an $E_{\mathfrak{F}}$-group.*

(3) *$G = G' \rtimes \langle x \rangle$, where $\langle x \rangle$ is a Sylow p-subgroup for a prime $p \in \pi(G)$ and a Carter subgroup, $G' = G^{\mathfrak{F}}$ and $G' \rtimes \langle x^p \rangle \in \mathfrak{F}$.*

Example. Let $\mathfrak{F}$ be the formation of all groups with nilpotent derived subgroups. We use E_{p^n} to denote an elementary abelian group of order p^n for a prime p and a positive integer n and C_m is a cyclic group of order m for a positive integer m.

Consider a group G [9, SmallGroup ID (864,4670)] $G = (S_3 \times S_3 \times A_4) \rtimes C_2$. In G, a Sylow 3-subgroup $G_3 \simeq E_{3^3}$ is $\mathfrak{F}$-subnormal and a Sylow 2-subgroup $G_2 \simeq E_{2^4} \rtimes C_2$ is self-normalizing, but G_2 is not $\mathfrak{F}$-subnormal and is not $\mathfrak{F}$-abnormal. Every proper subgroup of G_2 is $\mathfrak{F}$-subnormal in G. In addition,

$$G^{\mathfrak{F}} = F(G) \simeq E_{3^2} \times E_{2^2} < G^{\mathfrak{N}} \simeq E_{3^2} \times A_4 < G' \simeq (E_{3^2} \times A_4) \rtimes C_2.$$

Thus, G belongs to the class of groups with $\mathfrak{F}$-subnormal or self-normalizing primary subgroups, and G does not belong to the class of groups with $\mathfrak{F}$-subnormal or $\mathfrak{F}$-abnormal primary subgroups.

4. Groups with absolutely $\mathfrak{F}$-subnormal or self-normalizing primary subgroups

A. F. Vasil'ev proposed [10] the following concept.

Let $\mathfrak{F}$ be a formation. A subgroup H of a group G is called absolutely $\mathfrak{F}$-subnormal in G if any subgroup L containing H is $\mathfrak{F}$-subnormal in G.

In view of [10, Corrolary 3.2], the following lemma holds.

Lemma 6. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups. A group G belongs to $\mathfrak{F}$ if and only if every primary cyclic subgroup of G is absolutely $\mathfrak{F}$-subnormal in G.*

Theorem 2. *Let $\mathfrak{F}$ be a subgroup-closed saturated formation containing all nilpotent groups. Every primary cyclic subgroup of a group $G \notin \mathfrak{F}$ is absolutely $\mathfrak{F}$-subnormal or self-normalizing if and only if G is a non-nilpotent group all of whose proper subgroups are primary; in particular, $G = G' \rtimes \langle x \rangle$, G' is an elementary abelian p-group for a prime $p \in \pi(G)$, $\langle x \rangle$ is a maximal subgroup of order q and a Carter subgroup of G for a prime $q \in \pi(G)$ and $q \neq p$.*

Proof. Assume that every primary cyclic subgroup of a group $G \notin \mathfrak{F}$ is absolutely $\mathfrak{F}$-subnormal or self-normalizing. Since $\mathfrak{N} \subseteq \mathfrak{F}$, clearly $G \notin \mathfrak{N}$. If every primary cyclic subgroup of G is absolutely $\mathfrak{F}$-subnormal in G, then in view of Lemma 6 $G \in \mathfrak{F}$, a contradiction. Consequently, for a prime $q \in \pi(G)$ there is a cyclic q-subgroup $Q = \langle x \rangle$ that is not absolutely $\mathfrak{F}$-subnormal in G. By the choice of G, Q is self-normalizing, and so Q is a Sylow subgroup and a Carter subgroup of G. According to [1, IV.2.6], there is a Hall q'-subgroup $G_{q'}$ such that $G = G_{q'} \rtimes Q$. Clearly, $G^{\mathfrak{F}} \leq G' \leq G$. Since $G/G^{\mathfrak{N}}$ is nilpotent and $QG^{\mathfrak{N}}/G^{\mathfrak{N}}$ is a Sylow subgroup of $G/G^{\mathfrak{N}}$, we conclude that $QG^{\mathfrak{N}}$ is normal in G. By the Frattini lemma,

$$G = N_G(Q)(QG^{\mathfrak{N}}) = QG^{\mathfrak{N}} = G_{q'} \rtimes Q.$$

Consequently, $G^{\mathfrak{N}} = G_{q'} = G'$.

Let A be a maximal subgroup of Q. By the choice of G, A is absolutely $\mathfrak{F}$-subnormal or self-normalizing in G. If A is self-normalizing, then A is a Carter subgroup, and A is conjugate to Q [6], a contradiction. Hence A is absolutely $\mathfrak{F}$-subnormal in G, and Q is absolutely $\mathfrak{F}$-subnormal in G, a contradiction. Therefore $|Q| = q$.

Suppose that Q is not a maximal subgroup of G. Then every maximal subgroup M of G contains a primary cyclic r-subgroup R, $r \neq q$. If R is self-normalizing, then R is a Carter subgroup of G and R is conjugate to Q [6], a contradiction with $r \neq q$. Therefore R is absolutely $\mathfrak{F}$-subnormal in G, and M is $\mathfrak{F}$-subnormal in G. Thus every maximal subgroup of G is $\mathfrak{F}$-subnormal, and $G \in \mathfrak{F}$ by Lemma 4, a contradiction. Hence Q is a maximal subgroup, and G' is a minimal normal subgroup of G. Since G is soluble in view of [1, IV.7.4], G' is an elementary abelian p-group for a prime p, $p \neq q$.

Conversely, assume that $G = G' \rtimes \langle x \rangle$, G' is an elementary abelian p-group for a prime $p \in \pi(G)$, $\langle x \rangle$ is a maximal subgroup of order q and a Carter subgroup of G for a prime $q \in \pi(G)$ and $q \neq p$. Let A be a primary cyclic subgroup of G. If $\pi(A) = q$, then A is a Carter subgroup and $A = N_G(A)$. If $\pi(A) = p$, then $A \leq G' \in \mathfrak{A}$. Suppose that H is a proper subgroup of G such that $A \leq H$. Hence H is subnormal in G, and H is $\mathfrak{F}$-subnormal in G in view of [3, Lemma 1.11]. So A is absolutely $\mathfrak{F}$-subnormal in G. □

REFERENCES

1. Huppert B. *Endliche Gruppen* I. Berlin: Springer-Verl., 1967, 793 p.
2. Skiba A. N. On some results in the theory of finite partially soluble groups. *Commun. Math. Stat.*, 2016, vol. 4, no. 3, pp. 281–309. doi: 10.1007/s40304-016-0088-z .
3. Monakhov V. S. Finite groups with abnormal and $\mathfrak{U}$-subnormal subgroups. *Sib. Math. J.*, 2016, vol. 57, no. 2, pp. 352–363. doi: 10.1134/S0037446616020178 .
4. Monakhov V. S., Sokhor I. L. Finite groups with formation subnormal primary subgroups. *Sib. Math. J.*, 2017, vol. 58, no. 4, pp. 663–671. doi: 10.1134/S0037446617040127 .
5. Ballester-Bolinches A., Ezquerro L. M. *Classes of finite groups.* Dordrecht: Springer-Verl., 2006, 381 p.
6. Vdovin E. P. Carter subgroups of finite groups. *Sib. Adv. Math.*, 2009, vol. 19, no. 1, pp. 24–74. doi: 10.3103/S1055134409010039 .
7. Monakhov V. S. Schmidt subgroups, their existence and some applications. *Proceedings of Ukrainian Mathematical Congress*–2001, Inst. Mat. NAN Ukrainy Publ., Kyiv, 2002, pp. 81–90 (in Russian).
8. Semenchuk V.N. Soluble $\mathfrak{F}$-radical formations. *Math. Notes.*, 1996, vol. 59, no. 2, pp. 261–266 (in Russian).
9. The GAP Group: GAP — Groups, Algorithms, and Programming. Ver. 4.11.0 released on 29 February 2020. Available at: http://www.gap-system.org .
10. Vasil'ev A. F., Melchenko A. G. Finite groups with absolutely formationally subnormal Sylow subgroups. *Probl. Fiz. Math. Tekh.*, 2019, no. 4 (41), pp. 44–50 (in Russian).

Received October 19, 2020
Revised January 15, 2021
Accepted January 25, 2021

Irina Leonidovna Sokhor, Can. Phys.-Math. Sci., Doc., Brest State A. S. Pushkin University, 224000 Brest, Belarus, e-mail: irina.sokhor@gmail.com .

Cite this article as: I. L. Sokhor. Continuation of the theory of $E_{\mathfrak{F}}$-groups, *Trudy Instituta Matematiki i Mekhaniki URO RAN*, 2021, vol. 27, no. 1, pp. 268–272 .