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ON THE CONVERGENCE OF MINIMIZERS AND MINIMUM VALUES IN VARIATIONAL PROBLEMS WITH POINTWISE FUNCTIONAL CONSTRAINTS IN VARIABLE DOMAINS

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We consider a sequence of convex integral functionals $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ and a sequence of weakly lower semicontinuous and, in general, non-integral functionals $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$, where $\{\Omega_s\}$ is a sequence of domains in $\mathbb{R}^n$ contained in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) and $p > 1$. Along with this, we consider a sequence of closed convex sets $V_s = \{v \in W^{1,p}(\Omega_s) : M_s(v) \leqslant 0$ a.e. in $\Omega_s\}$, where M_s is a mapping from $W^{1,p}(\Omega_s)$ to the set of all functions defined on Ω_s. We describe conditions under which minimizers and minimum values of the functionals $F_s + G_s$ on the sets V_s converge to a minimizer and the minimum value of a functional on the set $V = \{v \in W^{1,p}(\Omega) : M(v) \leqslant 0$ a.e. in $\Omega\}$, where M is a mapping from $W^{1,p}(\Omega)$ to the set of all functions defined on Ω. In particular, for our convergence results, we require that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ and the sequence $\{F_s\}$ Γ-converges to a functional defined on $W^{1,p}(\Omega)$. In so doing, we focus on the conditions on the mappings M_s and M which, along with the corresponding requirements on the involved domains and functionals, ensure the convergence of solutions of the considered variational problems. Such conditions have been obtained in our recent work, and, in the present paper, we advance in studying them.

Keywords: variational problem, integral functional, pointwise functional constraint, minimizer, minimum value, Γ-convergence, strong connectedness, variable domains.

А. А. Ковалевский. О сходимости минимизантов и минимальных значений в вариационных задачах с поточечно функциональными ограничениями в переменных областях.

Рассмотрены последовательность выпуклых интегральных функционалов $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ и последовательность слабо полунепрерывных снизу и, вообще говоря, не интегральных функционалов $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$, где $\{\Omega_s\}$ — последовательность областей в $\mathbb{R}^n$, содержащихся в ограниченной области $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$), и $p > 1$. Наряду с этим рассмотрена последовательность замкнутых выпуклых множеств $V_s = \{v \in W^{1,p}(\Omega_s) : M_s(v) \leqslant 0$ п.в. в $\Omega_s\}$, где M_s — отображение $W^{1,p}(\Omega_s)$ во множество всех функций, определенных на Ω_s. Описаны условия, при которых минимизанты и минимальные значения функционалов $F_s + G_s$ на множествах V_s сходятся к минимизанту и минимальному значению некоторого функционала на множестве $V = \{v \in W^{1,p}(\Omega) : M(v) \leqslant 0$ п.в. в $\Omega\}$, где M — отображение $W^{1,p}(\Omega)$ во множество всех функций, определенных на Ω. В частности, требуется, чтобы последовательность пространств $W^{1,p}(\Omega_s)$ была сильно связана с пространством $W^{1,p}(\Omega)$ и последовательность $\{F_s\}$ Γ-сходилась к функционалу, определенному на $W^{1,p}(\Omega)$. При этом основное внимание уделено условиям на отображения M_s и M, которые вместе с соответствующими требованиями на участвующие области и функционалы обеспечивают сходимость решений рассматриваемых вариационных задач. Такие условия были получены в нашей недавней работе, и в настоящей статье мы продвинулись в их изучении.

Ключевые слова: вариационная задача, интегральный функционал, поточечно функциональное ограничение, минимизант, минимальное значение, Γ-сходимость, сильная связанность, переменные области.

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Introduction

One of the interesting questions in the theory of multidimensional homogenization is the study of the convergence of sequences of solutions of constrained minimization problems and variational inequalities (see, e.g., [1–5], where problems with explicit pointwise unilateral and bilateral constraints were considered). This study is closely related to the use of the notions of Γ-convergence of functionals and G-convergence of operators (for these notions in simple cases, see, e.g., [6; 7]).

Recently, in [8], we have described a large enough class of pointwise functional (in general, implicit) constraints for which the convergence of solutions of the corresponding variational problems is essentially defined by the Γ-convergence of the considered functionals and some general properties of the involved variable domains. In the present paper, we continue the study of the variational problems considered in [8].

Speaking in more detail, as in [8], we consider a sequence of convex integral functionals $F_s : W^{1,p}(\Omega_s) \to \mathbb{R}$ and a sequence of weakly lower semicontinuous and, in general, non-integral functionals $G_s : W^{1,p}(\Omega_s) \to \mathbb{R}$, where $\{\Omega_s\}$ is a sequence of domains in $\mathbb{R}^n$ contained in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) and $p > 1$. Along with this, we consider a sequence of closed convex sets

$$V_s = \{v \in W^{1,p}(\Omega_s) : M_s(v) \leqslant 0 \text{ a.e. in } \Omega_s\},$$

where M_s is a mapping from $W^{1,p}(\Omega_s)$ to the set of all functions defined on Ω_s. We describe conditions under which minimizers and minimum values of the functionals $F_s + G_s$ on the sets V_s converge to a minimizer and the minimum value of a functional on the set

$$V = \{v \in W^{1,p}(\Omega) : M(v) \leqslant 0 \text{ a.e. in } \Omega\},$$

where M is a mapping from $W^{1,p}(\Omega)$ to the set of all functions defined on Ω. In particular, for our convergence results, we require that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ and the sequence $\{F_s\}$ Γ-converges to a functional defined on $W^{1,p}(\Omega)$. In so doing, we focus on the conditions on the mappings M_s and M which, along with the corresponding requirements on the involved domains and functionals, ensure the convergence of solutions of the considered variational problems. Such conditions were obtained in [8]. In the present paper, we give two new conditions that together are sufficient for the fulfillment of an important condition on the mappings M_s and M established in [8].

The structure of this paper is as follows. In Section 1, we formulate necessary assumptions and definitions and recall our previous results related to the topic and used in further considerations. In Section 2, we state and prove our main result and give two theorems that follow from this result and the main theorems in [8]. Finally, in Section 3, we first give an example where the mappings defining the considered sets of constraints satisfy all the required conditions, and then we give two examples related to the verification of the conditions stated in the first example.

1. Preliminaries

Let $n \in \mathbb{N}$, $n \geqslant 2$, let Ω be a bounded domain in $\mathbb{R}^n$, and let $p > 1$. Let $\{\Omega_s\}$ be a sequence of domains in $\mathbb{R}^n$ contained in Ω.

It is easy to see that if $v \in W^{1,p}(\Omega)$ and $s \in \mathbb{N}$, then $v|_{\Omega_s} \in W^{1,p}(\Omega_s)$.

For every $s \in \mathbb{N}$, let $q_s\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega_s)$ be the mapping such that, for every function $v \in W^{1,p}(\Omega)$, $q_s v = v|_{\Omega_s}$.

Definition 1. We say that the sequence of domains Ω_s exhausts the domain Ω if, for every increasing sequence $\{m_j\} \subset \mathbb{N}$, we have

$$\operatorname{meas}\Big(\Omega \setminus \bigcup_{j=1}^{\infty} \Omega_{m_j}\Big) = 0.$$

Definition 2. We say that the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous operators $l_s\colon W^{1,p}(\Omega_s) \to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded and, for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$, we have $q_s(l_s v) = v$ a.e. in Ω_s.

We denote by $\mathcal{H}$ the set of all sequences $\{v_s\}$ such that, for every $s \in \mathbb{N}$, $v_s \in W^{1,p}(\Omega_s)$.

Definition 3. We say that the sequence $\{v_s\} \in \mathcal{H}$ is bounded if the sequence of norms $\|v_s\|_{W^{1,p}(\Omega_s)}$ is bounded.

For every function $v \in W^{1,p}(\Omega)$, we denote by $\mathcal{H}_0(v)$ the set of all sequences $\{v_s\} \in \mathcal{H}$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

It is easy to see that if $v \in W^{1,p}(\Omega)$, then $\{q_s v\} \in \mathcal{H}_0(v)$ and the sequence $\{q_s v\}$ is bounded.

Definition 4. For every $s \in \mathbb{N}$, let $I_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$, and let $I \colon W^{1,p}(\Omega) \to \mathbb{R}$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:

(a) for every function $v \in W^{1,p}(\Omega)$, there exists a sequence $\{w_s\} \in \mathcal{H}_0(v)$ such that $I_s(w_s) \to I(v)$;

(b) for every function $v \in W^{1,p}(\Omega)$ and every sequence $\{v_s\} \in \mathcal{H}_0(v)$, we have $\liminf\limits_{s\to\infty} I_s(v_s) \geqslant I(v)$.

We pass to the consideration of functionals for which we study the convergence of minimizers and minimum values on sets of functions with pointwise functional constraints.

Let $c_1, c_2 > 0$, and, for every $s \in \mathbb{N}$, let $\mu_s \in L^1(\Omega_s)$ and $\mu_s \geqslant 0$ in Ω_s. We assume that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded.

For every $s \in \mathbb{N}$, let $f_s \colon \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for every $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s; for almost every $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$; for almost every $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$,

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x,\xi) \leqslant c_2|\xi|^p + \mu_s(x). \tag{1.1}$$

By the assumptions on the functions f_s and μ_s, for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$, the function $f_s(x, \nabla v)$ is summable on Ω_s.

For every $s \in \mathbb{N}$, let $F_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be the functional such that, for every function $v \in W^{1,p}(\Omega_s)$,

$$F_s(v) = \int\limits_{\Omega_s} f_s(x, \nabla v)dx.$$

By the assumptions on the functions f_s and μ_s, for every $s \in \mathbb{N}$, the functional F_s is convex and locally bounded. Therefore, for every $s \in \mathbb{N}$, the functional F_s is weakly lower semicontinuous.

Further, let $c_3, c_4 > 0$, and, for every $s \in \mathbb{N}$, let $G_s \colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly lower semicontinuous functional. We assume that, for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$,

$$G_s(v) \geqslant c_3\|v\|^p_{L^p(\Omega_s)} - c_4. \tag{1.2}$$

It is clear that, for every $s \in \mathbb{N}$, the functional $F_s + G_s$ is weakly lower semicontinuous.

We define

$$c_5 = \min\{c_1/n, c_3\}, \qquad c_6 = c_4 + \sup_{s\in\mathbb{N}} \|\mu_s\|_{L^1(\Omega_s)}.$$

By (1.1) and (1.2), for every $s \in \mathbb{N}$ and every $v \in W^{1,p}(\Omega_s)$, we have

$$(F_s + G_s)(v) \geqslant c_5\|v\|^p_{W^{1,p}(\Omega_s)} - c_6.$$

Further, for every $s \in \mathbb{N}$, we denote by $\mathcal{F}(\Omega_s)$ the set of all functions $v \colon \Omega_s \to \mathbb{R}$.

For every $s \in \mathbb{N}$, let $M_s \colon W^{1,p}(\Omega_s) \to \mathcal{F}(\Omega_s)$. We assume that the following conditions are satisfied:

(A_1) there exists a bounded sequence $\{\psi_s\} \in \mathcal{H}$ such that, for every $s \in \mathbb{N}$, $M_s(\psi_s) \leqslant 0$ a.e. in Ω_s;

(A_2) if $s \in \mathbb{N}$ and $v_m \to v$ strongly in $W^{1,p}(\Omega_s)$, then there exists an increasing sequence $\{m_j\} \subset \mathbb{N}$ such that $M_s(v_{m_j}) \to M_s(v)$ a.e. in Ω_s;

(A_3) if $s \in \mathbb{N}$, $v, w \in W^{1,p}(\Omega_s)$, and $\tau \in [0,1]$, then

$$M_s((1-\tau)v + \tau w) \leqslant (1-\tau)M_s(v) + \tau M_s(w) \text{ a.e. in } \Omega_s.$$

For every $s \in \mathbb{N}$, we define

$$V_s = \{v \in W^{1,p}(\Omega_s)\colon M_s(v) \leqslant 0 \text{ a.e. in } \Omega_s\}.$$

It follows from conditions (A_1)–(A_3) that, for every $s \in \mathbb{N}$, the set V_s is nonempty, closed, and convex.

By virtue of the specified properties of the functionals $F_s + G_s$ and the sets V_s and the known results on the existence of minimizers of functionals (see, for instance, [9, Chapter 3]), for every $s \in \mathbb{N}$, there exists a function in V_s minimizing the functional $F_s + G_s$ on the set V_s.

We denote by $\mathcal{F}(\Omega)$ the set of all functions $v\colon \Omega \to \mathbb{R}$.

Let $M\colon W^{1,p}(\Omega) \to \mathcal{F}(\Omega)$. We define

$$V = \{v \in W^{1,p}(\Omega)\colon M(v) \leqslant 0 \text{ a.e. in } \Omega\}.$$

For the sequel, we assume that the following conditions are satisfied:

(C_1) the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact;

(C_2) the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$;

(C_3) the sequence of domains Ω_s exhausts the domain Ω.

Using conditions (C_1)–(C_3), we proved in [8] the following proposition.

Proposition 1. *Assume that the following condition is satisfied:*

(B_1) *if $v \in W^{1,p}(\Omega)$, $\{v_s\}$ is a bounded sequence in $\mathcal{H}_0(v)$, and $\{\hat{s}_k\}$ is an increasing sequence in $\mathbb{N}$, then there exist an increasing sequence $\{\tilde{s}_j\} \subset \{\hat{s}_k\}$ and a sequence of nonnegative functions $\beta_j\colon \Omega \to \mathbb{R}$ such that $\beta_j \to 0$ a.e. in Ω and, for every $j \in \mathbb{N}$, $M_{\tilde{s}_j}(v_{\tilde{s}_j}) \geqslant M(v) - \beta_j$ a.e. in $\Omega_{\tilde{s}_j}$.*

Let $\{w_s\}$ be a bounded sequence in $\mathcal{H}$ such that, for every $s \in \mathbb{N}$, $w_s \in V_s$. Let $\{\bar{s}_k\}$ be an increasing sequence in $\mathbb{N}$. Then there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $w \in V$ such that $\|w_{s_j} - q_{s_j} w\|_{L^p(\Omega_{s_j})} \to 0$.

Remark 1. It follows from condition (A_1) and Proposition 1 that if condition (B_1) of Proposition 1 is satisfied, then the set V is nonempty.

Further, we assume that the following conditions are satisfied:

(C_4) there exists a functional $F\colon W^{1,p}(\Omega) \to \mathbb{R}$ such that the sequence $\{F_s\}$ Γ-converges to the functional F;

(C_5) there exists a functional $G\colon W^{1,p}(\Omega) \to \mathbb{R}$ such that, for every function $v \in W^{1,p}(\Omega)$ and every bounded sequence $\{v_s\} \in \mathcal{H}_0(v)$, we have $G_s(v_s) \to G(v)$.

Using conditions (C_4), (C_5), (A_1), and (A_3) along with Proposition 1, inequalities (1.1) and (1.2), and the boundedness of the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$, we proved in [8] the following results (see Theorems 4.1 and 4.2 and Remark 4.1 in [8]).

Theorem 1.1. *Assume that condition (B_1) of Proposition 1 is satisfied. In addition, assume that the following condition is satisfied:*

(B') *if $v \in W^{1,p}(\Omega)$ and $M(v) \leqslant 0$ a.e. in Ω, then there exists a sequence $\{w_s\} \in \mathcal{H}_0(v)$ such that $\limsup_{s\to\infty} F_s(w_s) \leqslant F(v)$ and, for every $s \in \mathbb{N}$, $M_s(w_s) \leqslant 0$ a.e. in Ω_s.*

For every $s \in \mathbb{N}$, let u_s be a function in V_s minimizing the functional $F_s + G_s$ on the set V_s, and let $\{\bar{s}_k\}$ be an increasing sequence in $\mathbb{N}$. Then there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $u \in V$ such that the function u minimizes the functional $F + G$ on the set V, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, and $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

Theorem 1.2. *Assume that condition* (B_1) *of Proposition* 1 *and condition* (B') *of Theorem* 1.1 *are satisfied. In addition, assume that the set V is convex and the functional G is strictly convex. For every $s \in \mathbb{N}$, let u_s be a function in V_s minimizing the functional $F_s + G_s$ on the set V_s. Then there exists a unique function $u \in V$ minimizing the functional $F+G$ on the set V and the following relations hold*: $\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0$ *and* $(F_s + G_s)(u_s) \to (F+G)(u)$.

In view of the importance of condition (B') of Theorem 1.1 for the study of the convergence of solutions of the considered variational problems, we are interested in finding other conditions ensuring its fulfillment. Thus, in the statement of the above mentioned Theorems 4.1 and 4.2 in [8], we used, instead of condition (B') of Theorem 1.1, an equivalent condition which was verified in some specific cases (see [8, Sect. 6]). In the next section, we give two new conditions that together are sufficient for the fulfillment of condition (B') of Theorem 1.1. In this connection, we prove the following proposition.

Proposition 2. *The functional F is convex and continuous.*

Proof. We define

$$c_7 = \sup_{s\in\mathbb{N}} \|\mu_s\|_{L^1(\Omega_s)}, \qquad c_8 = c_2 n^p + c_7.$$

Let $v, w \in W^{1,p}(\Omega)$, and let $\alpha \in [0,1]$. By condition (C_4), there exist sequences $\{v_s\} \in \mathcal{H}_0(v)$ and $\{w_s\} \in \mathcal{H}_0(w)$ such that

$$F_s(v_s) \to F(v), \qquad F_s(w_s) \to F(w). \tag{1.3}$$

We set $z = (1-\alpha)v + \alpha w$ and, for every $s \in \mathbb{N}$, we define $z_s = (1-\alpha)v_s + \alpha w_s$. It is easy to see that $z \in W^{1,p}(\Omega)$ and $\{z_s\} \in \mathcal{H}_0(z)$. Then, by condition ($\mathrm{C}_4$), we have

$$F(z) \leqslant \liminf_{s\to\infty} F_s(z_s). \tag{1.4}$$

In turn, in view of the convexity of the functionals F_s, for every $s \in \mathbb{N}$, we have

$$F_s(z_s) \leqslant (1-\alpha)F_s(v_s) + \alpha F_s(w_s).$$

This along with (1.3) and (1.4) implies that

$$F((1-\alpha)v + \alpha w) \leqslant (1-\alpha)F(v) + \alpha F(w).$$

We also note that, by condition (C_4), $F(v) \leqslant \liminf\limits_{s\to\infty} F_s(q_s v)$ and, by (1.1), for every $s \in \mathbb{N}$,

$$-c_7 \leqslant F_s(v_s), \qquad F_s(q_s v) \leqslant c_2 \int\limits_{\Omega_s} |\nabla(q_s v)|^p\, dx + c_7.$$

These facts and the first relation in (1.3) yield the inequality

$$-c_7 \leqslant F(v) \leqslant c_2 \int\limits_{\Omega} |\nabla v|^p\, dx + c_7.$$

It follows from the above that the functional F is convex and, for every $v \in W^{1,p}(\Omega)$,

$$-c_7 \leqslant F(v) \leqslant c_8 (1 + \|v\|_{W^{1,p}(\Omega)})^p. \tag{1.5}$$

Now, let $v, w \in W^{1,p}(\Omega)$. We define

$$\lambda = \frac{\|v-w\|_{W^{1,p}(\Omega)}}{1 + \|v\|_{W^{1,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)}}\,. \tag{1.6}$$

First, assume that $\lambda \neq 0$. Then $\lambda \in (0,1)$. Using the convexity of the functional F, we obtain

$$F(v) = F((1-\lambda)w + \lambda(w + \lambda^{-1}(v-w))) \leqslant (1-\lambda)F(w) + \lambda F(w + \lambda^{-1}(v-w)).$$

This along with (1.5) and (1.6) implies that

$$F(v) - F(w) \leqslant 2^p(c_7 + c_8)(1 + \|v\|_{W^{1,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)})^{p-1}\|v-w\|_{W^{1,p}(\Omega)}.$$

Obviously, the same estimate we have for the difference $F(w) - F(v)$. Thus,

$$|F(v) - F(w)| \leqslant 2^p(c_7 + c_8)(1 + \|v\|_{W^{1,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)})^{p-1}\|v-w\|_{W^{1,p}(\Omega)}.$$

If $\lambda = 0$, we have $v = w$ a.e. in Ω. Then $F(v) = F(w)$ and, therefore, the previous inequality holds. From the obtained result, we deduce that the functional F is continuous. □

2. Main result and related theorems

Our main result is the following proposition.

Proposition 3. *Assume that the following conditions are satisfied:*

(A′) *if $v \in W^{1,p}(\Omega)$ and $M(v) \leqslant 0$ a.e. in Ω, then there exist a sequence $\{b_k\} \subset W^{1,p}(\Omega)$ and a sequence $\{\varepsilon_k\} \subset (0,+\infty)$ such that $b_k \to v$ strongly in $W^{1,p}(\Omega)$ and, for every $k \in \mathbb{N}$, $M(b_k) \leqslant -\varepsilon_k$ a.e. in Ω;*

(A″) *if $v \in W^{1,p}(\Omega)$, $\varepsilon > 0$, and $M(v) \leqslant -\varepsilon$ a.e. in Ω, then there exists a sequence $\{y_s\} \in \mathcal{H}_0(v)$ such that $\limsup\limits_{s\to\infty} F_s(y_s) \leqslant F(v)$ and, for every $s \in \mathbb{N}$, $M_s(y_s) \leqslant 0$ a.e. in Ω_s.*

Then condition (B′) *of Theorem* 1.1 *is satisfied.*

Proof. Let $v \in W^{1,p}(\Omega)$, and let $M(v) \leqslant 0$ a.e. in Ω. By condition (A′), there exist a sequence $\{b_k\} \subset W^{1,p}(\Omega)$ and a sequence $\{\varepsilon_k\} \subset (0,+\infty)$ such that

$$b_k \to v \text{ strongly in } W^{1,p}(\Omega), \tag{2.1}$$

$$\forall k \in \mathbb{N} \quad M(b_k) \leqslant -\varepsilon_k \text{ a.e. in } \Omega.$$

Then, by condition (A″), for every $k \in \mathbb{N}$, there exists a sequence $\{y_s^{(k)}\} \in \mathcal{H}_0(b_k)$ such that $\limsup\limits_{s\to\infty} F_s(y_s^{(k)}) \leqslant F(b_k)$ and

$$\forall s \in \mathbb{N} \quad M_s(y_s^{(k)}) \leqslant 0 \text{ a.e. in } \Omega_s. \tag{2.2}$$

Hence, there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that, for every $k \in \mathbb{N}$ and every $t \in \mathbb{N}$, $t \geqslant m_k$,

$$\|y_t^{(k)} - q_t b_k\|_{L^p(\Omega_t)} \leqslant k^{-1}, \tag{2.3}$$

$$F_t(y_t^{(k)}) \leqslant F(b_k) + k^{-1}. \tag{2.4}$$

For every $t \in \mathbb{N}$, we define

$$t_* = \begin{cases} 1 & \text{if } t \leqslant m_1, \\ \max\{k \in \mathbb{N} : t > m_k\} & \text{if } t > m_1. \end{cases}$$

Thus, for every $t \in \mathbb{N}$, we have $t_* \in \mathbb{N}$. In addition, if $t \in \mathbb{N}$ and $t > m_1$, then $t > m_{t_*}$. Now, for every $t \in \mathbb{N}$, we define $w_t = y_t^{(t_*)}$. It is easy to see that, for every $t \in \mathbb{N}$, $w_t \in W^{1,p}(\Omega_t)$. Next, we fix an arbitrary $\varepsilon > 0$. By (2.1) and the continuity of the functional F, there exists $k_1 \in \mathbb{N}$ such that $k_1 \geqslant 2/\varepsilon$ and, for every $k \in \mathbb{N}$, $k \geqslant k_1$,

$$\|b_k - v\|_{L^p(\Omega)} \leqslant \varepsilon/2, \qquad F(b_k) \leqslant F(v) + \varepsilon/2. \tag{2.5}$$

Let $t \in \mathbb{N}$, $t > m_{k_1}$. Obviously, $t_* \geqslant k_1$. Then, by (2.5), we have

$$\|b_{t_*} - v\|_{L^p(\Omega)} \leqslant \varepsilon/2, \qquad F(b_{t_*}) \leqslant F(v) + \varepsilon/2. \tag{2.6}$$

In addition, taking into account that $t > m_{t_*}$ and $t_* \geqslant k_1 \geqslant 2/\varepsilon$, we deduce from (2.3) and (2.4) the inequalities $\|w_t - q_t b_{t_*}\|_{L^p(\Omega_t)} \leqslant \varepsilon/2$ and $F_t(w_t) \leqslant F(b_{t_*}) + \varepsilon/2$. These inequalities and inequalities (2.6) imply that $\|w_t - q_t v\|_{L^p(\Omega_t)} \leqslant \varepsilon$ and $F_t(w_t) \leqslant F(v) + \varepsilon$. It follows from the above that $\{w_s\} \in \mathcal{H}_0(v)$ and $\limsup\limits_{s\to\infty} F_s(w_s) \leqslant F(v)$. Finally, taking into account (2.2), we find that, for every $s \in \mathbb{N}$, $M_s(w_s) \leqslant 0$ a.e. in Ω_s. Thus, condition (B$'$) of Theorem 1.1 is satisfied. □

From Theorems 1.1 and 1.2 and Proposition 3, we deduce the following results.

Theorem 2.3. *Assume that condition* (B$_1$) *of Proposition* 1 *and conditions* (A$'$) *and* (A$''$) *of Proposition* 3 *are satisfied. For every $s \in \mathbb{N}$, let u_s be a function in V_s minimizing the functional $F_s + G_s$ on the set V_s, and let $\{\bar{s}_k\}$ be an increasing sequence in $\mathbb{N}$. Then there exist an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ and a function $u \in V$ such that the function u minimizes the functional $F + G$ on the set V, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, and $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.*

Theorem 2.4. *Assume that condition* (B$_1$) *of Proposition* 1 *and conditions* (A$'$) *and* (A$''$) *of Proposition* 3 *are satisfied. In addition, assume that the set V is convex and the functional G is strictly convex. For every $s \in \mathbb{N}$, let u_s be a function in V_s minimizing the functional $F_s + G_s$ on the set V_s. Then there exists a unique function $u \in V$ minimizing the functional $F + G$ on the set V and the following relations hold: $\|u_s - q_s u\|_{L^p(\Omega_s)} \to 0$ and $(F_s + G_s)(u_s) \to (F + G)(u)$.*

3. Examples

We first give an example of the mappings M_s and M satisfying conditions (A$_1$)–(A$_3$) stated in Section 1, condition (B$_1$) of Proposition 1, and conditions (A$'$) and (A$''$) of Proposition 3. A similar example related to the verification of a condition equivalent to condition (B$'$) of Theorem 1.1 was considered in [8, Sect. 6]. However, as compared with [8] (more precisely, with [8, Example 6.3]), in the example below, we use weaker assumptions on the involved obstacle functions.

Example 1. We assume that the following condition is satisfied:

$$\text{for every sequence of measurable sets } H_s \subset \Omega_s \text{ such that } \operatorname{meas} H_s \to 0, \quad \int_{H_s} \mu_s \, dx \to 0. \tag{3.1}$$

Let $\varphi\colon \Omega \to \mathbb{R}$, and, for every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$. Let $\{\tau_s\} \subset [0, +\infty)$, $\tau_s \to 0$, and, for every $s \in \mathbb{N}$, let $\alpha_s\colon \Omega \to \mathbb{R}$ be a nonnegative function. We assume that

$$\alpha_s \to 0 \ \text{ a.e. in } \Omega, \tag{3.2}$$

$$\forall s \in \mathbb{N} \qquad \varphi - \tau_s \leqslant \varphi_s \leqslant \varphi + \alpha_s \ \text{ a.e. in } \Omega_s. \tag{3.3}$$

Let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$, and, for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be a continuous convex functional. We assume that the following condition is satisfied:

($*$) for every function $v \in W^{1,p}(\Omega)$ and every bounded sequence $\{v_s\} \in \mathcal{H}_0(v)$, $\Phi_s(v_s) \to \Phi(v)$.

We note that, by the convexity of the functionals Φ_s and condition ($*$), the functional Φ is convex.

Let $h\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing convex function. We assume that the following conditions are satisfied:

(i) there exist $\psi \in W^{1,p}(\Omega)$ and $c > 0$ such that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω;

(ii) there exists a bounded sequence $\{\bar{\varphi}_s\} \in \mathcal{H}$ such that, for every $s \in \mathbb{N}$, $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s.

Now, for every $s \in \mathbb{N}$, let $M_s\colon W^{1,p}(\Omega_s) \to \mathcal{F}(\Omega_s)$ be the mapping such that, for every function $v \in W^{1,p}(\Omega_s)$,

$$M_s(v) = h(v) - \varphi_s + \Phi_s(v).$$

The mappings M_s satisfy conditions (A_1)–(A_3) stated in Section 1. Indeed, it follows from condition (ii) that the mappings M_s satisfy condition (A_1). The continuity of the function h and the functionals Φ_s imply that the mappings M_s satisfy condition (A_2). Using the convexity of the function h and the functionals Φ_s, we easily establish that the mappings M_s satisfy condition (A_3).

Further, let $M\colon W^{1,p}(\Omega) \to \mathcal{F}(\Omega)$ be the mapping such that, for every function $v \in W^{1,p}(\Omega)$,

$$M(v) = h(v) - \varphi + \Phi(v).$$

Using conditions (C_1)–(C_3), the continuity of the function h, condition $(*)$, and assumptions (3.2) and (3.3), in the same way as in [8, Example 6.3], we find that the mappings M_s and M satisfy condition (B_1) of Proposition 1.

Next, we show that the mapping M satisfies condition (A') of Proposition 3. Let $v \in W^{1,p}(\Omega)$, and let $M(v) \leqslant 0$ a.e. in Ω. For every $k \in \mathbb{N}$, we define

$$b_k = (1 - k^{-1})v + k^{-1}\psi, \qquad \varepsilon_k = ck^{-1}.$$

Obviously, $\{b_k\} \subset W^{1,p}(\Omega)$ and $\{\varepsilon_k\} \subset (0, +\infty)$. It is also clear that $b_k \to v$ strongly in $W^{1,p}(\Omega)$. We fix an arbitrary $k \in \mathbb{N}$. Using the convexity of the functional Φ and the function h, we obtain

$$M(b_k) \leqslant (1 - k^{-1})M(v) + k^{-1}M(\psi) \text{ in } \Omega. \tag{3.4}$$

Since $M(v) \leqslant 0$ a.e. in Ω and, by condition (i), $M(\psi) \leqslant -c$ a.e. in Ω, we deduce from (3.4) that $M(b_k) \leqslant -\varepsilon_k$ a.e. in Ω. Thus, the mapping M satisfies condition (A') of Proposition 3.

Finally, we show that the mappings M_s and M satisfy condition (A'') of Proposition 3. Let $v \in W^{1,p}(\Omega)$, let $\varepsilon > 0$, and let $M(v) \leqslant -\varepsilon$ a.e. in Ω. By condition (C_4), there exists a sequence $\{v_s\} \in \mathcal{H}_0(v)$ such that $F_s(v_s) \to F(v)$. For every $s \in \mathbb{N}$, we define

$$\lambda_s = (\|v_s - q_s v\|_{L^1(\Omega_s)} + 1/s)^{1/2}.$$

Since $\{v_s\} \in \mathcal{H}_0(v)$, we have

$$\lambda_s \to 0. \tag{3.5}$$

Now, for every $s \in \mathbb{N}$, we define

$$\bar{v}_s = \min\{v_s - \lambda_s, q_s v\}, \qquad E_s = \{v_s - q_s v \geqslant \lambda_s\}.$$

It is easy to see that $\{\bar{v}_s\} \in \mathcal{H}$ and, for every $s \in \mathbb{N}$,

$$\|\bar{v}_s - q_s v\|_{L^p(\Omega_s)} \leqslant \|v_s - q_s v\|_{L^p(\Omega_s)} + \lambda_s(\operatorname{meas}\Omega)^{1/p}, \qquad \operatorname{meas} E_s \leqslant \lambda_s.$$

Then, by the inclusion $\{v_s\} \in \mathcal{H}_0(v)$ and (3.5), we have $\{\bar{v}_s\} \in \mathcal{H}_0(v)$ and $\operatorname{meas} E_s \to 0$. The latter fact and condition (3.1) imply that

$$\int\limits_{E_s} \mu_s \, dx \to 0, \qquad \int\limits_{E_s} |\nabla v|^p \, dx \to 0. \tag{3.6}$$

Fixing an arbitrary $s \in \mathbb{N}$, by the definition of the function $\bar{v}_s$ and the set E_s, we obtain

$$F_s(\bar{v}_s) = F_s(v_s) - \int\limits_{E_s} f_s(x, \nabla v_s)dx + \int\limits_{E_s} f_s(x, \nabla(q_s v))dx.$$

Hence, using (1.1), we get

$$F_s(\bar{v}_s) \leqslant F_s(v_s) + 2\int\limits_{E_s} \mu_s\,dx + c_2 \int\limits_{E_s} |\nabla v|^p\,dx.$$

Then, taking into account (3.6) and the fact that $F_s(v_s) \to F(v)$, we conclude that

$$\limsup_{s\to\infty} F_s(\bar{v}_s) \leqslant F(v).$$

Using this inequality along with (1.1), the boundedness of the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$, and the inclusion $\{\bar{v}_s\} \in \mathcal{H}_0(v)$, we find that the sequence $\{\bar{v}_s\}$ is bounded. Therefore, by condition $(*)$, we have $\Phi_s(\bar{v}_s) \to \Phi(v)$. In view of this and the convergence $\tau_s \to 0$, there exists $\bar{s} \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N}, s \geqslant \bar{s}, \qquad \tau_s + |\Phi_s(\bar{v}_s) - \Phi(v)| \leqslant \varepsilon. \tag{3.7}$$

We define the sequence $\{y_s\}$ as follows:

$$y_s = \begin{cases} \bar{\varphi}_s & \text{if } s \leqslant \bar{s}, \\ \bar{v}_s & \text{if } s > \bar{s}. \end{cases}$$

It is clear that $\{y_s\} \in \mathcal{H}_0(v)$ and $\limsup_{s\to\infty} F_s(y_s) \leqslant F(v)$. We fix an arbitrary $s \in \mathbb{N}$. If $s \leqslant \bar{s}$, by condition (ii), we have $M_s(y_s) \leqslant 0$ a.e. in Ω_s. Now, let $s > \bar{s}$. By (3.3) and the inequality $M(v) \leqslant -\varepsilon$ a.e. in Ω, there exists a set $E \subset \Omega_s$ of measure zero such that, for every $x \in \Omega_s \setminus E$,

$$\varphi(x) \leqslant \varphi_s(x) + \tau_s, \qquad h(v(x)) - \varphi(x) + \Phi(v) \leqslant -\varepsilon. \tag{3.8}$$

We fix an arbitrary $x \in \Omega_s \setminus E$. Since $\bar{v}_s(x) \leqslant v(x)$ and the function h is nondecreasing, we have

$$h(\bar{v}_s(x)) \leqslant h(v(x)). \tag{3.9}$$

Using the equality $y_s = \bar{v}_s$ and (3.7)–(3.9), we obtain

$$\begin{aligned} M_s(y_s)(x) &= h(\bar{v}_s(x)) - \varphi_s(x) + \Phi_s(\bar{v}_s) \\ &\leqslant h(v(x)) - \varphi(x) + \Phi(v) + \varphi(x) - \varphi_s(x) + \Phi_s(\bar{v}_s) - \Phi(v) \\ &\leqslant \tau_s + |\Phi_s(\bar{v}_s) - \Phi(v)| - \varepsilon \leqslant 0. \end{aligned}$$

Hence, $M_s(y_s) \leqslant 0$ a.e. in Ω_s. Thus, for every $s \in \mathbb{N}$, $M_s(y_s) \leqslant 0$ a.e. in Ω_s. From the above considerations, we conclude that the mappings M_s and M satisfy condition (A$''$) of Proposition 3.

Remark 2. It should be noted in connection with the above example that, in [8, Example 6.3], we assumed that the corresponding functions φ and φ_s satisfy conditions (3.2) and (3.3) and the corresponding functionals Φ and Φ_s satisfy condition $(*)$. However, instead of conditions (i) and (ii), we assumed in [8, Example 6.3] that the following condition is satisfied:

$$\forall s \in \mathbb{N} \quad \varphi_s \geqslant h(0) + \Phi_s(\theta_s) + c \text{ a.e. in } \Omega_s, \tag{3.10}$$

where $c > 0$ and, for every $s \in \mathbb{N}$, θ_s is the zero function on Ω_s. It follows from conditions (3.2), (3.3), (3.10), and $(*)$ that the same functions and functionals satisfy conditions (i) and (ii). Indeed, we have $\{\theta_s\} \in \mathcal{H}$, the sequence $\{\theta_s\}$ is bounded, and, by condition (3.10), for every $s \in \mathbb{N}$,

$h(\theta_s) + \Phi_s(\theta_s) \leqslant \varphi_s$ a.e. in Ω_s. Thus, condition (ii) is satisfied. Next, by (3.2), there exists a set $E' \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E' \qquad \alpha_s(x) \to 0. \tag{3.11}$$

In addition, by (3.3) and (3.10), there exists a set $E'' \subset \Omega$ of measure zero such that

$$s \in \mathbb{N},\ x \in \Omega_s \setminus E'' \implies h(0) + \Phi_s(\theta_s) + c \leqslant \varphi(x) + \alpha_s(x). \tag{3.12}$$

Finally, for every $r \in \mathbb{N}$, we define

$$E_r = \Omega \setminus \bigcup_{s=r}^{\infty} \Omega_s,$$

and let E''' be the union of all sets E_r, $r \in \mathbb{N}$. By condition (C_3), we have $\operatorname{meas} E''' = 0$. Now, let $x \in \Omega \setminus (E' \cup E'' \cup E''')$. We fix an arbitrary $\delta > 0$. By (3.11), there exists $k_1 \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\ s \geqslant k_1, \qquad \alpha_s(x) \leqslant \delta. \tag{3.13}$$

We denote by θ the zero function on Ω. Since $\{\theta_s\} \in \mathcal{H}_0(\theta)$, by condition $(*)$, we have $\Phi_s(\theta_s) \to \Phi(\theta)$. Then there exists $k_2 \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\ s \geqslant k_2, \qquad \Phi(\theta) \leqslant \Phi_s(\theta_s) + \delta. \tag{3.14}$$

We define $k = \max\{k_1, k_2\}$. Obviously, $x \in \Omega \setminus E_k$. Therefore, there exists $s \in \mathbb{N}$, $s \geqslant k$, such that $x \in \Omega_s$. Thus, $x \in \Omega_s \setminus E''$. Then, by (3.12)–(3.14), we have $h(0) + \Phi(\theta) + c \leqslant \varphi(x) + 2\delta$. Consequently, $h(\theta) + \Phi(\theta) \leqslant \varphi - c$ a.e. in Ω. Thus, condition (i) is satisfied. However, in general, it does not follow from conditions (i) and (ii) that condition (3.10) is satisfied. In this connection, see Example 2 below.

Remark 3. We note that condition (ii) in Example 1 almost follows from other conditions in this example. Indeed, by condition (i) in Example 1, there exists a set $E \subset \Omega$ of measure zero such that

$$\forall x \in \Omega \setminus E \qquad h(\psi(x)) + \Phi(\psi) \leqslant \varphi(x) - c. \tag{3.15}$$

Moreover, by condition $(*)$ in Example 1, we have $\Phi_s(q_s\psi) \to \Phi(\psi)$. In view of this and the convergence $\tau_s \to 0$, there exists $\tilde{s} \in \mathbb{N}$ such that

$$\forall s \in \mathbb{N},\ s \geqslant \tilde{s}, \qquad \tau_s + |\Phi_s(q_s\psi) - \Phi(\psi)| \leqslant c. \tag{3.16}$$

Let $s \in \mathbb{N}$, $s \geqslant \tilde{s}$. By (3.3), there exists a set $\tilde{E} \subset \Omega_s$ of measure zero such that

$$\forall x \in \Omega_s \setminus \tilde{E} \qquad \varphi(x) \leqslant \varphi_s(x) + \tau_s. \tag{3.17}$$

It follows from (3.15)–(3.17) that $h(q_s\psi) + \Phi_s(q_s\psi) \leqslant \varphi_s$ a.e. in Ω_s. Thus, setting, for every $s \in \mathbb{N}$, $\bar{\varphi}_s = q_s\psi$, we conclude that $\{\bar{\varphi}_s\} \in \mathcal{H}$, the sequence $\{\bar{\varphi}_s\}$ is bounded, and, for every $s \in \mathbb{N}$, $s \geqslant \tilde{s}$, $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ a.e. in Ω_s. This is what allows us to say that condition (ii) in Example 1 almost follows from other conditions in this example.

We now consider two examples where conditions $(*)$, (i), and (ii) stated in Example 1 are satisfied.

Example 2. Let $\varphi\colon \Omega \to \mathbb{R}$, and assume that the following condition is satisfied:

(i′) there exists a function $\bar{\varphi} \in W^{1,p}(\Omega)$ such that $\bar{\varphi} \leqslant \varphi$ a.e. in Ω.

For every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$. We assume that the following condition is satisfied:

(ii′) there exists a bounded sequence $\{\tilde{\varphi}_s\} \in \mathcal{H}$ such that, for every $s \in \mathbb{N}$, $\tilde{\varphi}_s \leqslant \varphi_s$ a.e. in Ω_s.

Next, for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be the functional such that, for every function $v \in W^{1,p}(\Omega_s)$, $\Phi_s(v) = \displaystyle\int_{\Omega_s} v\,dx$.

We assume that the following condition is satisfied:

($*'$) there exists a nonnegative bounded measurable function b on Ω such that, for every open cube $Q \subset \Omega$, we have $\operatorname{meas}(Q \cap \Omega_s) \to \int_Q b\, dx$.

By this condition, we have

$$\forall v \in L^1(\Omega) \qquad \int\limits_{\Omega_s} v\, dx \to \int\limits_{\Omega} bv\, dx. \tag{3.18}$$

Now, let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$ be the functional such that $\forall v \in W^{1,p}(\Omega)$ $\Phi(v) = \int_\Omega bv\, dx$. By (3.18), the functionals Φ_s and Φ satisfy condition ($*$) in Example 1.

Next, let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that, for every $t \in \mathbb{R}$, $h(t) = t$. We show that the functions h, φ, and φ_s and the functionals Φ_s and Φ satisfy conditions (i) and (ii) in Example 1. By condition (i$'$), there exists a function $\bar\varphi \in W^{1,p}(\Omega)$ such that $\bar\varphi \leqslant \varphi$ a.e. in Ω. We fix $c > 0$ such that

$$\int\limits_{\Omega} b\bar\varphi\, dx \leqslant c \int\limits_{\Omega} b\, dx \tag{3.19}$$

and define $\psi = \bar\varphi - c$. Using the inequality $\bar\varphi \leqslant \varphi$ a.e. in Ω and (3.19), we find that $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ a.e. in Ω. Thus, the functions h and φ and the functional Φ satisfy condition (i) in Example 1. Further, by condition (ii$'$), there exists a bounded sequence $\{\tilde\varphi_s\} \in \mathcal{H}$ such that, for every $s \in \mathbb{N}$, $\tilde\varphi_s \leqslant \varphi_s$ a.e. in Ω_s. For every $s \in \mathbb{N}$, we define

$$t_s = \frac{1}{\operatorname{meas}\Omega_s} \int\limits_{\Omega_s} \tilde\varphi_s\, dx.$$

Using the Hölder inequality, we find that

$$\forall s \in \mathbb{N} \qquad |t_s| (\operatorname{meas}\Omega_s)^{1/p} \leqslant \|\tilde\varphi_s\|_{L^p(\Omega_s)}. \tag{3.20}$$

Now, for every $s \in \mathbb{N}$, we define $\bar\varphi_s = \tilde\varphi_s - |t_s|$. Obviously, $\{\bar\varphi_s\} \in \mathcal{H}$. In addition, the boundedness of the sequence $\{\tilde\varphi_s\}$ and (3.20) imply that the sequence $\{\bar\varphi_s\}$ is bounded. It is also easy to see that, for every $s \in \mathbb{N}$, $h(\bar\varphi_s) + \Phi_s(\bar\varphi_s) \leqslant \varphi_s$ a.e. in Ω_s. Thus, the functions h and φ_s and the functionals Φ_s satisfy condition (ii) in Example 1.

Finally, assuming that the functions h and φ_s and the functionals Φ_s satisfy condition (3.10), we find that, for every $s \in \mathbb{N}$, $\varphi_s > 0$ a.e. in Ω_s. However, in general, this is not true. Thus, in general, it does not follow from conditions (i) and (ii) in Example 1 that condition (3.10) is satisfied.

Example 3. Let $\varphi\colon \Omega \to \mathbb{R}$ be a nonnegative function, and, for every $s \in \mathbb{N}$, let $\varphi_s\colon \Omega_s \to \mathbb{R}$ be a nonnegative function. In addition, for every $s \in \mathbb{N}$, let $\Phi_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be the functional such that, for every function $v \in W^{1,p}(\Omega_s)$, $\Phi_s(v) = \int_{\Omega_s} |v|^p\, dx$. We assume that condition ($*'$) in Example 2 is satisfied, and let $\Phi\colon W^{1,p}(\Omega) \to \mathbb{R}$ be the functional such that

$$\forall v \in W^{1,p}(\Omega) \qquad \Phi(v) = \int\limits_{\Omega} b|v|^p\, dx.$$

By (3.18), the functionals Φ_s and Φ satisfy condition ($*$) in Example 1.

Next, let $h\colon \mathbb{R} \to \mathbb{R}$ be the function such that, for every $t \in \mathbb{R}$, $h(t) = t$. We show that the functions h, φ, and φ_s and the functionals Φ_s and Φ satisfy conditions (i) and (ii) in Example 1. We fix $c > 0$ such that

$$c^{p-1} \int\limits_{\Omega} b\, dx \leqslant 2^{-p}, \tag{3.21}$$

and let $\psi\colon \Omega \to \mathbb{R}$ be the function such that, for every $x \in \Omega$, $\psi(x) = -2c$. Using (3.21), we find that $h(\psi) + \Phi(\psi) \leqslant -c$ in Ω. Therefore, in view of the nonnegativity of the function φ, we have $h(\psi) + \Phi(\psi) \leqslant \varphi - c$ in Ω. Thus, the functions h and φ and the functional Φ satisfy condition (i) in Example 1. Further, for every $s \in \mathbb{N}$, let $\bar{\varphi}_s$ be the zero function on Ω_s. Obviously, $\{\bar{\varphi}_s\} \in \mathcal{H}$ and the sequence $\{\bar{\varphi}_s\}$ is bounded. Moreover, by the nonnegativity of the functions φ_s, for every $s \in \mathbb{N}$, $h(\bar{\varphi}_s) + \Phi_s(\bar{\varphi}_s) \leqslant \varphi_s$ in Ω_s. Thus, the functions h and φ_s and the functionals Φ_s satisfy condition (ii) in Example 1.

In conclusion, we note that the fulfillment of conditions (C_1)–(C_5) is discussed, for instance, in [5].

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