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FINITE TOTALLY k-CLOSED GROUPS[1]

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For a positive integer k, a group G is said to be totally k-closed if in each of its faithful permutation representations, say on a set Ω, G is the largest subgroup of $\mathrm{Sym}(\Omega)$ which leaves invariant each of the G-orbits in the induced action on $\Omega\times\cdots\times\Omega=\Omega^k$. We prove that every finite abelian group G is totally $(n(G)+1)$-closed, but is not totally $n(G)$-closed, where $n(G)$ is the number of invariant factors in the invariant factor decomposition of G. In particular, we prove that for each $k\geq 2$ and each prime p, there are infinitely many finite abelian p-groups which are totally k-closed but not totally $(k-1)$-closed. This result in the special case $k=2$ is due to Abdollahi and Arezoomand. We pose several open questions about total k-closure.

Keywords: permutation group, k-closure, totally k-closed group.

Д. Чуриков, Ш. Прегер. Конечные вполне k-замкнутые группы.

Для натурального числа k группа G называется вполне k-замкнутой, если в каждом из ее точных подстановочных представлений, например, на множестве Ω группа G является наибольшей подгруппой $\mathrm{Sym}(\Omega)$, оставляющей на месте как множество каждую G-орбиту индуцированного действия на $\Omega\times\cdots\times\Omega=\Omega^k$. Доказано, что любая конечная абелева группа G вполне $(n(G)+1)$-замкнута, но не вполне $n(G)$-замкнута, где $n(G)$ — количество инвариантных множителей в разложении G на инвариантные множители. В частности, доказано, что для каждого натурального числа $k\geq 2$ и для каждого простого числа p существует бесконечно много конечных абелевых p-групп, которые вполне k-замкнуты, но не вполне $(k-1)$-замкнуты. В частном случае $k=2$ этот результат был получен Абдоллахи и Арезумандом. Поставлено несколько открытых вопросов о вполне k-замкнутых группах.

Ключевые слова: группа подстановок, k-замыкание, вполне k-замкнутая группа.

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1. Introduction

In 1969 Wielandt [7, Definition 5.3] introduced, for each positive integer k, the concept of the k-closure of a permutation group G on a set Ω. The *k-closure* $G^{(k),\Omega}$ of G is the set of all $g\in\mathrm{Sym}(\Omega)$ (permutations of Ω) such that g leaves invariant each G-orbit in the induced G-action on ordered k-tuples from Ω. The k-closure $G^{(k),\Omega}$ is a subgroup of $\mathrm{Sym}(\Omega)$ containing G [7, Theorem 5.4], and a permutation group G is said to be *k-closed* if $G^{(k),\Omega}=G$. Different faithful permutation representations of the same group G may have quite different k-closures. For example, the symmetric group S_3 acts faithfully and intransitively on $\{1,2,3,4,5\}$ with orbits $\{1,2,3\}$ and $\{4,5\}$, and in this action its 2-closure is $S_3\times C_2$; while S_3 is 2-closed in its natural action on $\{1,2,3\}$.

In 2016, D. F. Holt (see [8]) suggested a stronger concept independent of the permutation representation, and this was studied first by Abdollahi and Arezoomand in [1] in the case $k=2$. For a positive integer k, a group G is said to be *totally k-closed* if $G^{(k),\Omega}=G$ whenever G is faithfully represented as a permutation group on Ω. The only totally 1-closed group is the trivial group consisting of a single element (see Remark 2.3), while Abdollahi and Arezoomand [1, Theorem 2] showed that a finite nilpotent group is totally 2-closed if and only if it is cyclic, or it is a direct product of a generalised quaternion group and a cyclic group of odd order. Here we consider larger values of k.

[1]This paper is based on the results of the 2020 Ural Workshop on Group Theory and Combinatorics.

For a permutation group $G \le \mathrm{Sym}(\Omega)$, and $k \ge 2$, Wielandt [7, Theorem 5.8] proved that

$$G \le G^{(k),\Omega} \le G^{(k-1),\Omega}. \tag{1.1}$$

Thus if G is totally $(k-1)$-closed, then it is automatically totally k-closed. Moreover $G = G^{(k),\Omega}$ for sufficiently large k, since by [7, Theorem 5.12], this holds whenever there exist $k-1$ points $\alpha_1, \dots, \alpha_{k-1} \in \Omega$ such that the only element of G fixing each α_i is the identity. The inclusion (1.1) does suggest that the family of totally k-closed groups might be larger than that of totally $(k-1)$-closed groups. We show that this is the case, even for abelian groups.

Theorem 1.1. *Let k be an integer with $k \ge 2$. Then, for each prime p, there are infinitely many finite abelian p-groups which are totally k-closed but not totally $(k-1)$-closed.*

The result of Abdollahi and Arezoomand shows that the finite totally 2-closed abelian groups are precisely the cyclic groups. It turns out, also for larger values of k, that the total k-closure property for abelian groups is linked with the numbers of cyclic direct factors in their direct decompositions. A study of these decompositions leads to useful bounds, from which we deduce Theorem 1.1.

According to the fundamental theorem for finite abelian groups, each nontrivial finite abelian group G can be written as a direct product $G = H_1 \times \cdots \times H_n$, for some $n \ge 1$, such that each $H_i \cong \mathbb{Z}_{d_i}$, $d_1 > 1$, and $d_i | d_{i+1}$ for $1 \le i < n$. The integer n and the d_i are uniquely determined by G, up to the order of the factors. The H_i are called the *invariant factors* of G, and we write $n(G) := n$ for the number of invariant factors. We also have the primary decomposition of G as $G = \prod_{p \in \pi(G)} G_p$, where $\pi(G)$ is the set of primes dividing $|G|$ and G_p is the (unique) Sylow p-subgroup of G. It is straightforward to see that $n(G) = \max_{p \in \pi(G)} n(G_p)$. Our main result is the following theorem, from which we deduce Theorem 1.1.

Theorem 1.2. *Let G be a finite abelian group with $|G| > 1$. Then G is totally $(n(G)+1)$-closed, but is not totally $n(G)$-closed.*

The following auxiliary assertion on the k-closure of the direct product of abelian permutation p-groups may be of independent interest. It is proved in Section 2, and is used in Section 3 to reduce the proof of Theorem 1.2 to the case of p-groups. For its statement it is convenient to use $\mathrm{Syl}(G)$ to denote the set of all Sylow subgroups of a group G; if G is abelian, $\mathrm{Syl}(G)$ will consist of one Sylow p-subgroup for each prime $p \in \pi(G)$.

Theorem 1.3. *Let G be a finite abelian permutation group on a set Ω, and k an integer, $k \ge 2$. Then $G^{(k),\Omega} = \prod_{P \in \mathrm{Syl}(G)} P^{(k),\Omega}$.*

The results in our short paper serve to raise a number of open questions, and we record a few here. The first relates to Theorem 1.2. It would be interesting to have a generalisation of the classification by Abdollahi and Arezoomand [1, Theorem 2] of nilpotent totally 2-closed nilpotent groups for larger values of k.

Problem 1. *For $k > 2$ determine all finite nilpotent groups G that are totally k-closed.*

As we noted above, the symmetric group S_3 is not totally 2-closed. Indeed, it was shown by Abdollahi, Arezoomand and Tracey [2, Theorem B] that a finite soluble group is totally 2-closed if and only if it is nilpotent, hence known by [1, Theorem 2]. However it is not difficult to see that it is totally 3-closed, since in every faithful permutation representation of $G = S_3$ on a set Ω there must be a G-orbit of length 3 or 6, and the stabiliser in G of two points α, β from such an orbit is trivial. Hence by [7, Theorem 5.12], $G = G^{(3),\Omega}$. As a first step it would be interesting to know which other non-nilpotent soluble groups are totally 3-closed.

Problem 2. *Determine the finite soluble groups that are totally 3-closed.*

For some time it was believed that all finite totally 2-closed groups would be soluble, and it was somewhat surprising to discover[2] that exactly six of the sporadic simple groups are totally 2-closed, namely J_1, J_3, J_4, Ly, Th, M.

Problem 3. *Find all the the totally 3-closed sporadic simple groups. More generally, for each sporadic simple group G determine the least value of k such that G is totally k-closed.*

The classification of the finite nonabelian simple totally 2-closed groups is still not complete, and we refer the reader to the manuscript[2] in preparation by M. Arezoomand, M.A. Iranmanesh, C.E. Praeger, and G. Tracey for details of the status of this problem and other open questions about total 2-closure.

2. Preliminaries

In this section we give some background theory, and in particular we prove Theorem 1.3. First we state two results of Wielandt for convenience.

Theorem 2.1 (Wielandt, [7, Theorem 5.6]). *Let $G \leq \mathrm{Sym}(\Omega)$, let $k \geq 1$, and let $x \in \mathrm{Sym}(\Omega)$. Then $x \in G^{(k),\Omega}$ if and only if, for all $(\alpha_1, \dots, \alpha_k) \in \Omega^k$, there exists $g \in G$ such that $\alpha_i^x = \alpha_i^g$ for $i = 1, \dots, k$.*

Theorem 2.2 (Wielandt, [7, Theorem 5.12]). *Let $G \leq \mathrm{Sym}(\Omega)$ and $k \geq 1$, and suppose that $\alpha_1, \dots, \alpha_k \in \Omega$ such that $G_{\alpha_1 \dots \alpha_k} = 1$. Then $G^{(k+1),\Omega} = G$.*

Next we discuss total 1-closure.

Remark 2.3. Suppose that G is a finite totally 1-closed group. Consider the regular representation of G on $\Omega = G$. Since G is transitive on Ω it follows from Theorem 2.1 that $G^{(1),\Omega} = \mathrm{Sym}(\Omega)$. Thus, since G is totally 1-closed, it follows that $\mathrm{Sym}(\Omega) = G$ is regular, and hence $|G| \leq 2$. However, if $G = C_2$, then in the representation $G = \langle (12)(34) \rangle \leq \mathrm{Sym}(\Omega)$ on $\Omega = \{1,2,3,4\}$ we have $G^{(1),\Omega} = \langle (12), (34) \rangle \neq G$. Hence $G = 1$ is the only possibility.

For a prime $p \mid n$, the largest p-power divisor of n is denoted by n_p; if π is a set of prime divisors of n, then $n_\pi := \prod_{p \in \pi} n_p$ denotes the π-part of n. Recall that, for a finite group G, $\pi(G)$ is the set of prime divisors of $|G|$. For $p \in \pi(G)$, we denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G. For a subgroup $G \leq \mathrm{Sym}(\Omega)$ we denote by $\mathrm{Orb}(G)$ the set of G-orbits in Ω.

The proof of Theorem 1.3 is developed using ideas from [4]. First we present separately two lemmas as they are general results about finite nilpotent groups.

Lemma 2.4. *Let G be a finite nilpotent permutation group, let $p \in \pi(G)$, k be a positive integer, and $P \in \mathrm{Syl}_p(G)$. Let $\Delta_1, \dots, \Delta_k \in \mathrm{Orb}(P)$, $\Delta = \bigcup_{i=1}^k \Delta_i$, and L be the subgroup of G consisting of all elements fixing each Δ_i setwise. Then $L^\Delta = P^\Delta$.*

Proof. By the definition of L, the subgroup $P \leq L$, and hence $P^\Delta \leq L^\Delta$. We now prove the converse. Since G is nilpotent, we have $G = P \times H$, where H is the Hall p'-subgroup of G. Let $g \in L$, so $g = xy$ for some (unique) $x \in P$ and $y \in H$. Since $P \leq L$, we have $y = x^{-1}g \in L$.

We claim that $y^\Delta = 1$, or equivalently, that $y^{\Delta_i} = 1_{\Delta_i}$ for each $i = 1, \dots, k$. Since $y \in H \leq C_G(P)$ it follows that, for each i, y^{Δ_i} belongs to the centralizer Z_i of the transitive group $P^{\Delta_i} \leq \mathrm{Sym}(\Delta_i)$, which is semiregular by [6, Theorem 3.2]. In particular $|Z_i|$ divides $|\Delta_i|$ which is a p-power, so Z_i is a p-group. Consequently, y^{Δ_i} is a p-element. Since $y \in H$ and $|H|$ is coprime to p, this implies that $y^{\Delta_i} = 1$, for each i, and hence that $y^\Delta = 1$, proving the claim. Thus, $g^\Delta = (xy)^\Delta = x^\Delta y^\Delta = x^\Delta \in P^\Delta$, as required.

[2] Arezoomand M., Iranmanesh M.A., Praeger C.E., and Tracey G. Totally 2-closed finite simple groups, in preparation.

Lemma 2.5 [4, Lemma 2.4]. *Let $G \leq \mathrm{Sym}(\Omega)$, where $n = |\Omega|$ and $\pi \subseteq \pi(G)$. Suppose that G is transitive and nilpotent, and let H be a Hall π-subgroup of G. Then*

(1) *the size of every H-orbit is equal to n_π, and*

(2) *G acts on $\mathrm{Orb}(H)$; moreover, the kernel of this action is equal to H.*

Proof of Theorem 1.3

Let G be a finite abelian permutation group on a set Ω, and let $k \geq 2$. Then by [7, Theorem 5.8] and [7, Exercise 5.26], $G^{(k),\Omega}$ is abelian, and $\pi(G^{(k),\Omega}) = \pi(G)$. Let $p \in \pi(G)$, and let P and Q be the (unique) Sylow p-subgroups of G and $G^{(k),\Omega}$ respectively.

Claim 1. $P \leq P^{(k),\Omega} \leq Q$, and $\mathrm{Orb}(P) = \mathrm{Orb}(Q)$.

Proof of Claim 1. By [7, Theorem 5.8] and [7, Exercise 5.28], the group $P^{(k),\Omega}$ is a p-group, and hence $P \leq P^{(k),\Omega} \leq Q$. It remains to prove that each P-orbit is a Q-orbit. Let Δ be a P-orbit, and let Γ be the G-orbit containing Δ. By (1.1), $G \leq G^{(k),\Omega} \leq G^{(1),\Omega}$, and hence $G^{(k),\Omega}$ has the same orbits as G in Ω. Thus Γ is also a $G^{(k),\Omega}$-orbit, and hence the Q-orbit Δ' containing Δ satisfies $\Delta \subseteq \Delta' \subseteq \Gamma$. The induced permutation groups G^Γ and $(G^{(k),\Omega})^\Gamma$ are both transitive and abelian, so applying Lemma 2.5 to each of these groups with Hall subgroups P^Γ, Q^Γ, respectively, yields $|\Delta| = |\Gamma|_p = |\Delta'|$. Thus $\Delta = \Delta'$, and Claim 1 is proved. □

Claim 2. $P^{(k),\Omega} = Q$.

Proof of Claim 2. Let $(\alpha_1, \ldots, \alpha_k) \in \Omega^k$, and $g \in Q$. By Theorem 2.1, there exists $h \in G$ such that

$$(\alpha_1, \ldots, \alpha_k)^g = (\alpha_1, \ldots, \alpha_k)^h.$$

For each $i = 1 \ldots k$, let Δ_i be the Q-orbit containing α_i. Then by Claim 1, each Δ_i is also a P-orbit. Since $P \trianglelefteq G$, the group G permutes the P-orbits, and for each i, since $\alpha_i^h = \alpha_i^g \in \Delta_i$, it follows that h fixes each Δ_i setwise. Thus h lies in the subgroup L of Lemma 2.4, and setting $\Delta = \bigcup_{i=1}^k \Delta_i$, it follows from Lemma 2.4 that $h^\Delta = u^\Delta$ for some $u \in P$. Thus

$$(\alpha_1, \ldots, \alpha_k)^g = (\alpha_1, \ldots, \alpha_k)^h = (\alpha_1, \ldots, \alpha_k)^u.$$

Since such an element $u \in P$ exists for each k-tuple of points and each $g \in Q$, it follows from Theorem 2.1 that $g \in P^{(k),\Omega}$. Thus $Q \leq P^{(k),\Omega}$, and the reverse inclusion holds by Claim 1. □

Now we complete the proof of Theorem 1.3. Since $G^{(k),\Omega}$ is abelian, $G^{(k),\Omega}$ is the direct product of its Sylow subgroups. Further, for each $p \in \pi(G)$ it follows from Claim 2 that the unique Sylow p-subgroup of $G^{(k),\Omega}$ is $P^{(k),\Omega}$, where P is the unique Sylow p-subgroup of G. □

3. Proof of the main results

Recall the definition of $n(G)$ given in Section 1 for a finite abelian group G. We also set $N(G) := \sum_{p\in\pi(G)} n(G_p)$. If $G \leq \mathrm{Sym}(\Omega)$ then the *base size* $b(G,\Omega)$ of G is the smallest integer b for which there exist $\alpha_1, \ldots, \alpha_b \in \Omega$ such that $G_{\alpha_1 \ldots \alpha_b} = 1$. Such a set $\alpha_1, \ldots, \alpha_b$ is called a *base* of G. Note that, by Theorem 2.2, $G = G^{(b+1),\Omega}$, where $b = b(G,\Omega)$.

Lemma 3.1. *Let G be a finite abelian group and suppose that G has a faithful permutation representation on a finite set Ω. Then $b(G,\Omega) \leq N(G)$, and equality holds for some Ω.*

Proof. Let $G = \prod_{p\in\pi(G)} G_p$ with $\pi(G)$ the set of primes dividing $|G|$, and G_p the Sylow p-subgroup of G, for $p \in \pi(G)$. Then, by the definition of $N(G)$ and the $n(G_p)$, G has a direct decomposition $G = H_1 \times \cdots \times H_n$, with each H_i nontrivial and cyclic of prime power order, and $n = N(G)$. For each i, H_i acts regularly on $\Omega_i := H_i$ by (right) multiplication, and G acts faithfully on $\Omega := \cup_{i=1}^n \Omega_i$ (where H_j acts trivially on Ω_i for $i \neq j$). Thus the G-orbits in Ω are the sets Ω_i, and for each i the subgroup H_i acts nontrivially only on the orbit Ω_i. Thus each base must contain a point from each of the G-orbits. It follows that the base size equals $N(G)$ for this faithful permutation representation of G.

Now consider an arbitrary faithful permutation representation of G, that is, suppose that $G \leq \mathrm{Sym}(\Omega)$. We prove by induction on $N(G)$ that G has base size at most $N(G)$. Now $H_1 = \langle h_1 \rangle \cong \mathbb{Z}_{p^a}$, for some prime p and positive integer a, and as G acts faithfully on Ω there exists $\alpha \in \Omega$ which is not fixed by $h_1^{p^{a-1}}$. This implies that $G_\alpha \cap H_1 = 1$. If $N(G) = 1$ then $G = H_1$ is a cyclic p-group, and $G_\alpha = 1$, so $\{\alpha\}$ is a base. Assume now that $N(G) \geq 2$ and that the assertion holds for groups X with $N(X) < N(G)$. Since $G_\alpha \cap H_1 = 1$, we have $G_\alpha \cong (G_\alpha H_1)/H_1 \leq G/H_1 \cong \prod_{i=2}^n H_i$ so $N(G_\alpha) \leq n-1 = N(G)-1$, and hence by induction, G_α has a base $\alpha_1, \dots, \alpha_s$ in $\Omega \setminus \{\alpha\}$ with $s \leq N(G)-1$. Then $\alpha_1, \dots, \alpha_s, \alpha$ is a base for G in Ω, and the result follows by induction.

We now prove Theorem 1.2 in the case of p-groups. The second part of the lemma is proved using a construction developed from ideas in the book of Chen and Ponomarenko [3, Proposition 2.2.26]. An element $\tau \in \mathrm{Sym}(\Omega)$ is called a *cycle* if it is not the identity and has exactly one cycle of length greater than 1 in its disjoint cycle representation; the length of this cycle is denoted $|\tau|$. Two cycles are said to be *independent* if the sets of points they move are disjoint.

Lemma 3.2. *Let G be a finite abelian p-group with $|G| > 1$. Then G is totally $(n(G)+1)$-closed, but is not totally $n(G)$-closed.*

Proof. Since G is an abelian p-group, $N(G) = n(G)$. By Lemma 3.1, if G is faithfully represented as a subgroup of $\mathrm{Sym}(\Omega)$, then $b := b(G, \Omega) \leq n(G)$, and by Theorem 2.2, $G = G^{(b+1),\Omega}$. It follows from (1.1) that $G = G^{(n(G)+1),\Omega}$. Since this holds for all faithful permutation representations of G, G is totally $(n(G)+1)$-closed.

As discussed in Section 1, $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \ldots \times \mathbb{Z}_{d_n}$, with $d_1 > 1$, $d_i | d_{i+1}$ for $1 \leq i < n$, and $n = n(G)$. Let Ω be a set of size $d_1 + \sum_{i=1}^n d_i$, and let $\tau_0, \tau_1, \dots, \tau_n \in \mathrm{Sym}(\Omega)$ be pairwise independent cycles on Ω such that $|\tau_0| = d_1$, and $|\tau_i| = d_i$ for $i = 1 \dots n$. Let $H_1 = \langle \tau_0 \tau_1 \rangle$ and $H_i = \langle \tau_0^{-1} \tau_i \rangle$ for $i = 2 \dots n$, and let $H = \langle H_1, \dots, H_n \rangle$. We claim that $H \cong G$. Indeed, the groups H_i commute, and an easy proof by induction on n shows that $H_i \cap \langle H_1, \dots, H_{i-1}, H_{i+1}, \dots, H_n \rangle = 1$, for $i = 1 \dots n$. Thus $H = H_1 \times \ldots \times H_n$, with $H_i \cong \mathbb{Z}_{d_i}$ for $i = 1 \dots n$, proving the claim.

Now we will use Theorem 2.1 to show that $\tau_0 \in H^{(n),\Omega}$. Let $(\alpha_1, \dots, \alpha_n) \in \Omega^n$, and for $i = 0, \dots, n$, let Δ_i denote the set of points of Ω moved by τ_i, so that $\{\Delta_0, \dots, \Delta_n\}$ is the set of H-orbits in Ω. Since H has $n+1$ nontrivial orbits, there exists $k \in \{0, 1, \dots, n\}$ such that $\Delta_k \cap \{\alpha_1, \dots, \alpha_n\} = \varnothing$. Define a permutation τ as follows:

$$\tau = \begin{cases} 1, & \text{if } k = 0, \\ \tau_0 \tau_k^{-1}, & \text{if } 1 \leq k \leq n. \end{cases}$$

By definition, $\tau \in H$. If $\tau = 1$, then both τ and τ_0 fix each of the α_i so $(\alpha_1, \dots, \alpha_n)^{\tau_0} = (\alpha_1, \dots, \alpha_n)^\tau$. On the other hand, if $\tau = \tau_0 \tau_k^{-1}$ for some k, then τ and τ_0 induce the same permutation on $\Omega \setminus \Delta_k$, and again we have $(\alpha_1, \dots, \alpha_n)^{\tau_0} = (\alpha_1, \dots, \alpha_n)^\tau$. Thus, by Theorem 2.1, $\tau_0 \in H^{(n),\Omega}$. By the construction, $\tau_0 \notin H$, and hence $H \neq H^{(n),\Omega}$. Thus G is not totally n-closed.

Remark 3.3. Theorem 1.1 follows from Lemma 3.2 since, for each integer $k \geq 2$, there are infinitely many finite abelian p-groups with k invariant factors. □

Finally we prove Theorem 1.2 for an arbitrary finite abelian group G with $|G| > 1$. Suppose that G is faithfully represented on a set Ω. Since $n(G) = \max_{p \in \pi(G)} n(G_p)$, every Sylow subgroup G_p of G is $(n(G)+1)$-closed by Lemma 3.2, and hence, by Theorem 1.3, we have $G^{(n(G)+1),\Omega} = G$. Thus G is totally $(n(G)+1)$-closed.

Set $n := n(G)$. If $n = 1$ then, since $|G| > 1$, it follows from Remark 2.3 that G is not totally 1-closed. Thus we may assume that $n \geq 2$. Now $n(G) = \max_{p \in \pi(G)} n(G_p)$, and hence we have

$n = n(G_q)$ for some $q \in \pi(G)$. By Lemma 3.2, G_q is not totally n-closed, so there exists a set Ω_q such that G_q acts faithfully on Ω_q and $G^{(n),\Omega_q} \neq G_q$. There is nothing further to prove if $G = G_q$ so we may assume that $|\pi(G)| \geq 2$. For each $p \in \pi(G) \setminus \{q\}$, let $\Omega_p = G_p$, and consider G_p acting regularly on Ω_p by right multiplication. Thus G acts faithfully on $\Omega := \cup_{p\in\pi(G)}\Omega_p$. Since $n \geq 2$, it follows from Theorem 1.3 that

$$G^{(n),\Omega} = \prod_{p\in\pi(G)} (G_p)^{(n),\Omega_p} = (G_q)^{(n),\Omega_q} \times \prod_{\substack{p\in\pi(G)\\ p\neq q}} (G_p)^{(n),\Omega_p},$$

which is not equal to G, because $(G_q)^{(n),\Omega_q} > G_q$ and for every $p \in \pi(G), p \neq q$ the group G_p is n-closed as a regular group. Thus, G is not totally n-closed, and the proof of Theorem 1.2 is complete. □

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