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CONVERGENCE OF EIGENELEMENTS IN A STEKLOV TYPE BOUNDARY VALUE PROBLEM FOR THE LAMÉ OPERATOR

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A Steklov type boundary value problem is studied for the Lamé operator in a half-strip with a small hole. On the lateral boundaries of the half-strip and on the boundary of the small hole, homogeneous Dirichlet boundary conditions are specified, and the Steklov spectral condition is specified on the base of the half-strip. A theorem is proved on the convergence of the eigenelements of this problem to the solution of the limit problem (in the half-strip without a hole) as a small parameter $\varepsilon > 0$ characterizing the diameter of the hole tends to zero. To prove the theorem, we introduce the space of infinitely differentiable vector functions with a finite Dirichlet integral over the half-strip and prove a number of auxiliary statements. The Dirichlet integral for a vector function is defined as the sum of the Dirichlet integrals of the components. Among the auxiliary statements, in particular, it is proved that the weak convergence in the metric of the introduced space of a sequence of functions defined on the half-strip implies the convergence of their restrictions to the base of the half-strip in the metric of the space L_2. In addition, it is proved that, for the solutions of Steklov type boundary value problems for the Lamé operator in a half-strip with a small hole, the weak convergence of the restrictions to the base of the half-strip implies strong convergence in the same domain. For each value of the parameter ε, an operator is defined for the restriction of the solutions of the considered boundary value problems to the base of the half-strip. The convergence of the sequence of inverses of the restriction operators is also proved as $\varepsilon \to 0$. A physical interpretation of the solution of the singularly perturbed boundary value problem considered in the paper is that this solution simulates the deformation vector of an elastic homogeneous isotropic medium filling a two-dimensional region with a small hole. The Lamé equation is an equilibrium equation under which a stationary state of an elastic medium in the form of a plate can be maintained. The Dirichlet boundary conditions at the lateral boundaries of the half-strip and at the boundary of the small hole can be interpreted as a rigid fixation of the elastic plate. The spectral Steklov condition specified at the base of the strip is a complex elastic fixation. The eigenvalues and the corresponding eigenvector functions of the boundary value problem characterize the possible natural vibrations of the elastic plate.

Keywords: boundary value problem, Steklov spectral condition, Lamé operator, eigenelements, small parameter.

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