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ON A REFINEMENT OF MARCINKIEWICZ–ZYGMUND TYPE INEQUALITIES

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The main goal of this paper is to verify a refined Marcinkiewicz–Zygmund type inequality with a quadratic error term

$$\frac{1}{2}\sum_{j=0}^{nm-1}(x_{j+1}-x_{j-1})w(x_j)|t_n(x_j)|^q=(1+O(m^{-2}))\int\limits_{-\pi}^{\pi}w(x)|t_n(x)|^q\,dx,\quad 2\le q<\infty,$$

where t_n is any trigonometric polynomial of degree at most n, $-\pi=x_0<x_1<\dots<x_{mn}=\pi$, $\max\limits_{0\le j\le mn-1}(x_{j+1}-x_j)=O\Big(\dfrac{1}{nm}\Big)$, $m,n\in\mathbb{N}$, and w is a Jacobi type weight. Moreover, the quadratic error term $O(m^{-2})$ is shown to be sharp, in general. In addition, similar results are given for $q=\infty$ and in the multivariate case.

Keywords: multivariate polynomials, Marcinkiewicz–Zygmund, Bernstein, and Schur type inequalities, discretization of L^p norm, doubling and Jacobi type weights.

А. Кроо. Об уточнении неравенств типа Марцинкевича — Зигмунда.

В статье доказано уточненное неравенство типа Марцинкевича — Зигмунда с квадратичным остаточным членом

$$\frac{1}{2}\sum_{j=0}^{nm-1}(x_{j+1}-x_{j-1})w(x_j)|t_n(x_j)|^q=(1+O(m^{-2}))\int\limits_{-\pi}^{\pi}w(x)|t_n(x)|^q\,dx,\quad 2\le q<\infty,$$

где t_n — произвольный тригонометрический полином степени не больше n, $-\pi=x_0<x_1<\dots<x_{mn}=\pi$, $\max\limits_{0\le j\le mn-1}(x_{j+1}-x_j)=O\Big(\dfrac{1}{nm}\Big)$, $m,n\in\mathbb{N}$ и w — вес типа Якоби. Также показано, что квадратичный остаточный член $O(m^{-2})$ в общем случае является точным. Аналогичные результаты получены при $q=\infty$ и в случае многих переменных.

Ключевые слова: многочлены от нескольких переменных, неравенства типа Марцинкевича — Зигмунда, Бернштейна и Шура, дискретизация нормы L^p, веса типа Якоби и с условием удвоения.

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1. Introduction

The starting point for the present paper is the classical Marcinkiewicz—Zygmund result [11] stating that for any univariate trigonometric polynomial t_n of degree at most n and every $1\le q<\infty$ we have

$$\int|t_n|^q\sim\frac{1}{n}\sum_{s=0}^{2n}\Big|t_n\Big(\frac{2\pi s}{2n+1}\Big)\Big|^q,\tag{1}$$

where the constants depend only on q. This equivalence relation is an effective tool used for the discretization of the L_q norms of trigonometric polynomials, which is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, and scattered data interpolation; see for instance [10] for a survey on the univariate Marcinkiewicz—Zygmund type inequalities. An important generalization of (1) for the so-called *doubling weights* was given by Mastroianni and Totik [12]. Recall that a nonnegative integrable weight w on $[-\pi,\pi]$ is called doubling if with certain $L>0$ depending only on the weight

$$\int_{2I} w \le L \int_{I} w, \quad I \subset [-\pi,\pi]$$

for any interval I and $2I$ being its double with the same midpoint. In particular, all generalized Jacobi type weights satisfy the doubling property. Then as shown in [12] there exists an integer $m_w \in \mathbb{N}$ (depending only on the weight) such that whenever $m \ge m_w$ we have with $x_j := \dfrac{\pi j}{mn}$, $0 \le j \le 2mn$

$$B_m \sum_{0\le j\le 2mn-1} c_j |t_n(x_j)|^q \le \int_{-1}^{1} |t_n|^q w \le A_m \sum_{0\le j\le 2mn-1} c_j |t_n(x_j)|^q \quad \forall\, t_n \in T_n, \tag{2}$$

where T_n stands for the set of real trigonometric polynomials of degree at most n, $c_j := \displaystyle\int_{x_j-1/n}^{x_j+1/n} w(t)\,dt$, $0 \le j \le mn$, and $A_m, B_m > 0$ depend only on m and w.

Subsequently, in [4] similar Marcinkiewicz—Zygmund type inequalities were given for various multivariate domains, in particular polytopes, cones, spherical sectors, tori, etc.

A crucial feature of estimates (2) consists in the fact that the constants A_m, B_m are independent of the degree n of the trigonometric polynomials. On the other hand, it is natural to expect that the optimal constants must satisfy the relations $A_m, B_m \to 1$ as $m \to \infty$. Indeed, a careful examination of the proofs given in [4] reveals that estimates similar to (2) hold with $A_m, B_m = 1 + O\Big(\dfrac{1}{m}\Big)$. This leads to the question of the sharp rate of convergence $A_m, B_m \to 1$ as $m \to \infty$. The main goal of the present paper is to verify that when $q \ge 2$ Marcinkiewicz—Zygmund type inequalities analogous to (2) hold with

$$A_m, B_m = 1 + O\Big(\frac{1}{m^2}\Big). \tag{3}$$

This *quadratic error term* for the constants A_m, B_m will be verified for both algebraic and trigonometric polynomials of one and several variables. Moreover, we will also show that in general the error term $O\Big(\dfrac{1}{m^2}\Big)$ is the *best possible*. The problem of finding the correct asymptotics of A_m, B_m is similar to the so-called "Marcinkiewicz problem with ϵ" raised in [3, p. 5], which corresponds to the situation when A_m, $B_m = 1 + \epsilon$.

The analogue of Marcinkiewicz—Zygmund type inequalities for $q = \infty$ is the notion of *admissible meshes* or *norming sets*, see [2; 6]. Admissible meshes $Y_n \subset K$, $n \in \mathbb{N}$, are discrete point sets satisfying with some $c > 0$ depending only on K

$$\|p\|_K \le c\|p\|_{Y_n} \quad \forall\, p \in P_n^d, \quad \forall\, n \in \mathbb{N}, \tag{4}$$

where P_n^d stands for the set of algebraic polynomials in d variables of total degree at most n, and $\|p\|_D$ denotes the usual sup norm on the compact set $D \subset \mathbb{R}^d$. If, in addition, card $Y_n \sim n^d$, then the admissible mesh is called optimal. In [8] it was shown that star-like C^2-domains and convex polytopes in $\mathbb{R}^d$ possess optimal meshes, while in [9] the existence of optimal meshes was verified for every convex domain on the plane.

Recently in [13] it was proved that for a C^2 star-like domain $K \subset \mathbb{R}^d$ there exist optimal meshes of cardinality $(mn)^d$ so that (4) holds with $c = 1 + O\Big(\dfrac{1}{m^2}\Big)$, $m \in \mathbb{N}$. This exhibits a quadratic error term when $q = \infty$. However, the question of sharpness of this quadratic error term when $q = \infty$ was not addressed in [13]. We will fill in this gap below.

This paper is organized as follows. First we will prove the sharpness of $c = 1 + O\Big(\dfrac{1}{m^2}\Big)$ in (4) for every compact set $K \subset \mathbb{R}^d$ and each mesh with card $Y_n \sim (mn)^d$ (Theorem 1). Then we will verify

Marcinkiewicz—Zygmund type inequalities (2) with quadratic error terms (3) for both algebraic and trigonometric polynomials, see Theorems 2, 3 and Corollaries 1, 2. In addition, it will be proved that, in general, the error term $O\Big(\frac{1}{m^2}\Big)$ is the best possible (Theorem 4 below). Finally, Theorem 5 gives similar estimates on the disc and this allows to proceed to more general multidimensional domains.

Let $B(\mathbf{x},\mathbf{r})$ stand for the closed ball in $\mathbb{R}^d$ of radius r and center $\mathbf{x}$. Furthermore, we denote by R_K and r_K the radii of the smallest ball containing K and of the largest ball embedded into K, respectively. Moreover, $\rho_K := \frac{r_K}{R_K}$ is the so called *distortion constant* of K.

Theorem 1. *Consider any compact set $K \subset \mathbb{R}^d$ with nonempty interior. Then for each subset $Y_n \subset K$ of cardinality $(mn)^d \geq 2,\ m, n \in \mathbb{N}$ there exists $Q \in P^d_{2n}$ such that*

$$\|Q\|_K \geq \Big(1 + \frac{\rho_K^2}{2m^2}\Big)\|Q\|_{Y_n}. \tag{5}$$

If in addition K is convex or convex and central symmetric then the above estimate holds with ρ_K replaced by $\frac{1}{d}$ or $\frac{1}{\sqrt{d}}$, respectively.

In the proof of Theorem 1 we will use the following "needle polynomial" type result. Denote by $T_n(x) := \cos(n \arccos x)$ the classical Chebyshev polynomial.

Lemma 1. *For any $0 < h < 1$ consider the even univariate polynomial*

$$q_n(x) := T_n\Big(\frac{2x^2 - h^2 - 1}{1 - h^2}\Big) \in P^1_{2n}.$$

Then $|q_n(x)| \leq 1$ for every $h \leq |x| \leq 1$ and

$$|q_n(0)| \geq 1 + 2n^2h^2. \tag{6}$$

P r o o f. Clearly, $-1 \leq \frac{2x^2 - h^2 - 1}{1 - h^2} \leq 1$ for $h \leq |x| \leq 1$ and thus $|q_n(x)| \leq 1$ on this set. Moreover, by the well-known representation of Chebyshev polynomials

$$T_n(x) = \frac{1}{2}\Big((x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n\Big)$$

it follows that

$$|q_n(0)| = T_n\Big(\frac{1 + h^2}{1 - h^2}\Big) = \frac{1}{2}\Big(\frac{1 + h^2}{1 - h^2} + \frac{2h}{1 - h^2}\Big)^n + \frac{1}{2}\Big(\frac{1 + h^2}{1 - h^2} - \frac{2h}{1 - h^2}\Big)^n = \frac{(1 + h)^{2n} + (1 - h)^{2n}}{2(1 - h^2)^n}.$$

Hence using the binomial formula

$$|q_n(0)| \geq \frac{1 + \binom{2n}{2}h^2}{(1 - h^2)^n} \geq \big(1 + n(2n - 1)h^2\big)(1 + h^2)^n \geq \big(1 + n(2n - 1)h^2\big)(1 + nh^2) \geq 1 + 2n^2h^2. \ \Box$$

P r o o f of Theorem 1. Consider any subset $Y_n \subset K$ of cardinality $(mn)^d$. Denote by

$$s_n := \max_{\mathbf{x}\in K} \min_{\mathbf{y}\in Y_n} |\mathbf{x} - \mathbf{y}|$$

the *fill distance* of Y_n in K. Then we clearly have $K \subset (\bigcup_{\mathbf{y}\in Y_n} B(\mathbf{y}, s_n))$, and hence denoting by m_d the Lebesgue measure of the unit ball in $\mathbb{R}^d$

$$r_K^d m_d \leq m(K) \leq (mn)^d s_n^d m_d,$$

where $m(K)$ is the Lebesgue measure of K. In addition, at least one of the above inequalities has to be strict yielding

$$r_K < mns_n. \tag{7}$$

Furthermore, by the definition of s_n there exists $\mathbf{x}_0 \in K$ such that for every $\mathbf{y} \in Y_n$ we have $|\mathbf{x}_0 - \mathbf{y}| \geq s_n$.

Now set

$$Q(\mathbf{x}) := q_n\Big(\frac{|\mathbf{x} - \mathbf{x}_0|}{2R_K}\Big),$$

where q_n is the needle polynomial of Lemma 1 of degree $2n$ with

$$h := \frac{r_K}{2mnR_K} = \frac{\rho_K}{2mn}.$$

Note that since q_n is an even univariate polynomial of degree $2n$ it follows that $Q \in P^d_{2n}$. Furthermore, since K is contained in a ball of radius R_K it clearly follows that the diameter of K is at most $2R_K$. Thus for every $\mathbf{y} \in Y_n$ we have by (7)

$$h = \frac{r_K}{2mnR_K} < \frac{s_n}{2R_K} \leq \frac{|\mathbf{y} - \mathbf{x}_0|}{2R_K} \leq 1.$$

Hence by Lemma 1 we have $\|Q\|_{Y_n} \leq 1$. Finally, by (6) we obtain

$$\|Q\|_K \geq |Q(\mathbf{x}_0)| = |q_n(0)| \geq 1 + 2n^2h^2 = 1 + \frac{\rho_K^2}{2m^2} \geq \Big(1 + \frac{\rho_K^2}{2m^2}\Big)\|Q\|_{Y_n}.$$

Now assume that K is convex. First it should be noted that definition (4) of norming sets of K is invariant under regular affine transformations of the domain. That is if Y_n is a norming set of K then for any regular affine map $T : \mathbb{R}^d \to \mathbb{R}^d$ the set $T(Y_n)$ is a norming set of $T(K)$ with the same norming constant c in (4). Secondly, by the John's maximal ellipsoidal theorem [7] there exists a unique ellipsoidal E_K of maximal volume and center $\mathbf{c}_K$ such that $E_K \subset K \subset \mathbf{c}_K + d(E_K - \mathbf{c}_K)$, where $E_K = T(B(\mathbf{0}, 1))$ with some regular affine map $T : \mathbb{R}^d \to \mathbb{R}^d$. Moreover, if in addition K is convex and central symmetric then the same holds true with $\sqrt{d}$ instead of d. Thus when K is convex we may assume without loss of generality that $B(\mathbf{0}, 1) \subset K \subset B(\mathbf{0}, d)$, or $B(\mathbf{0}, 1) \subset K \subset B(\mathbf{0}, \sqrt{d})$ in the central symmetric case. Therefore, if K is convex, or convex and central symmetric we can assume that $\rho_K = \dfrac{1}{d}$ or $\rho_K = \dfrac{1}{\sqrt{d}}$, respectively. Using these relations together with (5) completes the proof of the theorem. □

2. Refined Marcinkiewicz—Zygmund inequalities for univariate polynomials

Our next result refines the classical Marcinkiewicz—Zygmund inequality. It provides constants of order $1+O\Big(\dfrac{1}{m^2}\Big)$ in (1) when discretization is accomplished with mn nodes. A basic tool needed below is the L_q Bernstein inequality for trigonometric polynomials, see [5, p. 102], or [1]. It states that for every $t_n \in T_n$

$$\|t_n'\|_{L_q} \leq n\|t_n\|_{L_q}.$$

Theorem 2. *For any n, $N \in \mathbb{N}$, $q \geq 2$, $-\pi = x_0 < x_1 < \cdots < x_N = \pi$ and $t_n \in T_n$ we have*

$$\Big|\int_{-\pi}^{\pi} |t_n(x)|^q\,dx - \frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1} - x_{j-1})|t_n(x_j)|^q\Big| \leq \frac{(qnh_N)^2}{2}\int_{-\pi}^{\pi} |t_n(x)|^q\,dx, \tag{8}$$

where $h_N := \max_{0\leq j\leq N-1}(x_{j+1} - x_j)$, $x_{-1} := x_{N-1} - 2\pi$.

P r o o f. We will use the following elementary trapezoidal integration rule which can be easily verified by repeated integration by parts for any function f whose first derivative is absolutely continuous

$$\int_a^b f(x)\,dx = (b-a)\frac{f(a)+f(b)}{2} + \frac{1}{2}\int_a^b f''(x)(a-x)(b-x)\,dx. \tag{9}$$

We have by (9) setting $f_n(x) := |t_n(x)|^q$

$$\int_{-\pi}^{\pi} |t_n(x)|^q\,dx = \sum_{j=0}^{N-1}\int_{x_j}^{x_{j+1}} |t_n(x)|^q\,dx = \sum_{j=0}^{N-1}\Big(\frac{(x_{j+1}-x_j)(|t_n(x_j)|^q + |t_n(x_{j+1})|^q)}{2} + R_j\Big),$$

where

$$R_j := \frac{1}{2}\int_{x_j}^{x_{j+1}} f_n''(x)(x_j-x)(x_{j+1}-x)\,dx, \quad |R_j| \le \frac{h_N^2}{2}\int_{x_j}^{x_{j+1}} |f_n''(x)|\,dx.$$

It is easy to see that

$$\frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1}-x_j)\big(|t_n(x_j)|^q + |t_n(x_{j+1})|^q\big) = \frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1}-x_j)|t_n(x_j)|^q + \frac{1}{2}\sum_{j=1}^{N}(x_j-x_{j-1})|t_n(x_j)|^q$$

$$= \frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1}-x_{j-1})|t_n(x_j)|^q.$$

Thus, combining the above relations, we get

$$R := \int_{-\pi}^{\pi} |t_n(x)|^q\,dx - \frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1}-x_{j-1})|t_n(x_j)|^q = \sum_{j=0}^{N-1} R_j, \quad |R| \le \frac{h_N^2}{2}\int_{-\pi}^{\pi} |f_n''(x)|\,dx. \tag{10}$$

Since obviously

$$|f_n''(x)| \le q(q-1)|t_n(x)|^{q-2}(t_n')^2 + q|t_n(x)|^{q-1}|t_n''|,$$

it follows that

$$|R| \le \frac{h_N^2 q(q-1)}{2}\int_{-\pi}^{\pi} |t_n(x)|^{q-2}(t_n')^2\,dx + \frac{h_N^2 q}{2}\int_{-\pi}^{\pi} |t_n(x)|^{q-1}|t_n''|\,dx. \tag{11}$$

Using the Hölder inequality and the L_q-Bernstein inequality for trigonometric polynomials we obtain for the first integral in (11)

$$\int_{-\pi}^{\pi} |t_n(x)|^{q-2}(t_n')^2\,dx \le \|t_n\|_{L_q}^{q-2}\|t_n'\|_{L_q}^2 \le n^2\|t_n\|_{L_q}^q.$$

Likewise, for the second integral we have

$$\int_{-\pi}^{\pi} |t_n(x)|^{q-1}|t_n''|\,dx \le \|t_n\|_{L_q}^{q-1}\|t_n''\|_{L_q} \le n^2\|t_n\|_{L_q}^q.$$

Substituting the last two estimates in (11) we finally obtain $|R| \le \dfrac{h_N^2 q^2 n^2}{2}\|t_n\|_{L_q}^q$, which is the needed upper bound. □

Theorem 2 easily yields the next Marcinkiewicz—Zygmund type result for trigonometric polynomials which provides a remainder term of quadratic accuracy. We set $x_j := \dfrac{\pi j}{mn}$, $-mn \le j \le mn$, to be the $2mn$ equidistant points on the period. Then $h_m = \dfrac{\pi}{mn}$ in (8) yielding the next corollary.

Corollary 1. *For any* $2 \le q < \infty$, $m \ge 3q$, $n \in \mathbb{N}$, *and every* $t_n \in T_n$ *we have with some* $|c| \le \dfrac{\pi^2}{2}$

$$\frac{\pi}{mn} \sum_{j=-mn}^{mn-1} \Big| t_n\Big(\frac{\pi j}{mn}\Big)\Big|^q = \Big(1 + \frac{cq^2}{m^2}\Big) \int_{-\pi}^{\pi} |t_n(x)|^q \, dx.$$

Now we proceed to verifying weighted versions of Theorem 2. In order to prove weighted Marcinkiewicz—Zygmund type inequalities we need to recall certain L_q Bernstein and Schur type inequalities with doubling weights which can be found in [12]. It is shown in [12] that given any doubling weight w and $1 \le q < \infty$ there is a constant $c > 0$ depending only on the weight such that for every trigonometric polynomial of degree at most n

$$\int_{-\pi}^{\pi} |t_n'(x)|^q w \, dx \le cn^q \int_{-\pi}^{\pi} |t_n(x)|^q w \, dx. \tag{12}$$

Consider now a periodic Jacobi type weight

$$w(x) := w_0(x) \prod_{1 \le k \le s} |x - y_k|^{a_k}, \quad x,\ y_k \in [-\pi, \pi],$$

$$a_k \ge 1, \quad 1 \le k \le s, \quad s \in \mathbb{N}, \quad w_0 \in C^2[-\pi, \pi], \quad w_0 > 0. \tag{13}$$

The defect of this weight denoted by d_w is defined as $d_w := \max_k a_k$. The following Schur type inequality is given in [12]: for any doubling weight w^*, $1 \le q < \infty$, and every trigonometric polynomial of degree at most n

$$\int_{-\pi}^{\pi} |t_n(x)|^q w^* \, dx \le cn^{d_w} \int_{-\pi}^{\pi} |t_n(x)|^q ww^* \, dx, \tag{14}$$

where the constant $c > 0$ depends only on the weights.

Theorem 3. *Let* w *be a Jacobi type weight* (13)*. Then for any* $n, N \in \mathbb{N}$, $q \ge 2$, $-\pi = x_0 < x_1 < \dots < x_N = \pi$, *and* $t_n \in T_n$ *we have*

$$\frac{1}{2} \sum_{j=0}^{N-1} (x_{j+1} - x_{j-1}) w(x_j) |t_n(x_j)|^q = \big(1 + O(q^2 n^2 h_N^2)\big) \int_{-\pi}^{\pi} w(x) |t_n(x)|^q \, dx, \tag{15}$$

where $h_N := \max\limits_{0 \le j \le N-1} (x_{j+1} - x_j)$ *and the constant in the* $O(\dots)$ *term depends only on the weight. In particular, if* $N = nm$, $m \in \mathbb{N}$, *and* $h_N = O\Big(\dfrac{1}{nm}\Big)$ *then we get*

$$\frac{1}{2} \sum_{j=0}^{N-1} (x_{j+1} - x_{j-1}) w(x_j) |t_n(x_j)|^q = \big(1 + O(m^{-2})\big) \int_{-\pi}^{\pi} w(x) |t_n(x)|^q \, dx.$$

P r o o f. Evidently we can apply estimate (10) with $|t_n|^q$ replaced by $w(x)|t_n(x)|^q$, and in particular, $f_n(x) := w(x)|t_n(x)|^q$. Then in turn (11) can be replaced by the estimate

$$|R| \le q^2 h_N^2 \int_{-\pi}^{\pi} \big(w(t_n')^2 |t_n|^{q-2} + w|t_n|^{q-1}|t_n''| + |t_n|^{q-1}|t_n' w'| + |t_n|^q |w''| \big) \, dx. \tag{16}$$

Now we will estimate the four terms on the right hand side of (16) by repeated application of the Hölder inequality and weighted L_q Bernstein and Schur type inequalities.

Define

$$\|t_n\|_{L_q(w)}^q := \int_{-\pi}^{\pi} w(x)|t_n(x)|^q\, dx.$$

1st term. For the first term we use the Hölder inequality and L_q Bernstein inequality (12) with the doubling weight w. Then we have

$$\int_{-\pi}^{\pi} w(t_n')^2|t_n|^{q-2}\, dx = \int_{-\pi}^{\pi} \big(w^{(q-2)/q}|t_n|^{q-2}\big)w^{2/q}(t_n')^2\, dx \le \|t_n\|_{L_q(w)}^{q-2}\|t_n'\|_{L_q(w)}^2 \le cn^2\|t_n\|_{L_q(w)}^q.$$

2nd term. Again the Hölder inequality and (12) yield

$$\int_{-\pi}^{\pi} w|t_n|^{q-1}|t_n''|\, dx = \int_{-\pi}^{\pi} w^{(q-1)/q}|t_n|^{q-1}w^{1/q}|t_n''|\, dx \le \|t_n\|_{L_q(w)}^{q-1}\|t_n''\|_{L_q(w)} \le cn^2\|t_n\|_{L_q(w)}^q.$$

3rd term. This case is somewhat trickier. First we use the Hölder inequality to deduct

$$\int_{-\pi}^{\pi} |t_n|^{q-1}|t_n'w'|\, dx \le \Bigg(\int_{-\pi}^{\pi} |t_n|^q|w'|\, dx\Bigg)^{(q-1)/q}\Bigg(\int_{-\pi}^{\pi} |t_n'|^q|w'|\, dx\Bigg)^{1/q}. \tag{17}$$

Then for both integrals on the right hand of (17) we can use the Schur type inequality (14) with the doubling weight $|w'|$ and Jacobi type weight

$$w_1(x) := \prod_{1\le k\le s} |x-y_k|, \quad d_{w_1} = 1.$$

It can be easily verified that $|w'(x)|w_1(x) \le cw(x),\ x \in [-\pi,\pi]$, with some $c > 0$ depending only on the weight. Thus by (17) and (14)

$$\int_{-\pi}^{\pi} |t_n|^{q-1}|t_n'w'|\, dx \le c\Bigg(n\int_{-\pi}^{\pi} |t_n|^q|w'|w_1\, dx\Bigg)^{(q-1)/q}\Bigg(n\int_{-\pi}^{\pi} |t_n'|^q|w'|w_1\, dx\Bigg)^{1/q}$$

$$\le cn\Bigg(\int_{-\pi}^{\pi} |t_n|^q w\, dx\Bigg)^{(q-1)/q}\Bigg(\int_{-\pi}^{\pi} |t_n'|^q w\, dx\Bigg)^{1/q}.$$

Furthermore, by (12)

$$\Bigg(\int_{-\pi}^{\pi} |t_n'|^q w\, dx\Bigg)^{1/q} \le cn\|t_n\|_{L_q(w)}.$$

Clearly the last two upper bounds imply the following estimate of the 3rd term:

$$\int_{-\pi}^{\pi} |t_n|^{q-1}|t_n'w'|\, dx \le cn^2\|t_n\|_{L_q(w)}^{q-1}\|t_n\|_{L_q(w)} = cn^2\|t_n\|_{L_q(w)}^q.$$

4th term. This time we apply Schur type inequality (14) for the doubling weight $|w''|$ and Jacobi type weight

$$w_2(x) := \prod_{1\le k\le s} |x-y_k|^2, \quad d_{w_2} = 2.$$

Again it can be verified that $|w''(x)|w_2(x) \le cw(x),\ x \in [-\pi,\pi]$, with some $c > 0$ depending only on the weight. Then by (14)

$$\int_{-\pi}^{\pi} |t_n|^q|w''|\,dx \le cn^2 \int_{-\pi}^{\pi} |t_n|^q|w''|w_2\,dx \le cn^2 \int_{-\pi}^{\pi} w|t_n|^q\,dx = cn^2\|t_n\|^q_{L_q(w)}.$$

The above estimates verify an upper bound of order $n^2\|t_n\|^q_{L_q(w)}$ for each term on the right-hand side of (16) with a constant depending only on w. Hence summarizing we find from (16) that

$$|R| \le cq^2h_N^2n^2\|t_n\|^q_{L_q(w)}.$$

Evidently, this upper bound immediately implies the required estimate (15). □

Theorem 3 implies a Marcinkiewicz—Zygmund type result for univariate algebraic polynomials with a remainder term of quadratic accuracy for a Jacobi type weight (13) on $[-1,1]$. Indeed, the next corollary easily follows from the second relation in Theorem 3 by a standard trigonometric substitution $x = \cos y$.

Corollary 2. *Let $m, n \in \mathbb{N},\ 2 \le q < \infty$, and set $x_j := \cos\dfrac{\pi j}{mn},\ 1 \le j \le mn-1$. Then for every $p_n \in P_n^1$ and any Jacobi type weight w on $[-1,1]$ given as in (13) we have*

$$\frac{\pi}{mn}\sum_{j=1}^{mn-1}\sqrt{1-x_j^2}\,w(x_j)|p_n(x_j)|^q = \bigl(1+O(m^{-2})\bigr)\int_{-1}^{1} w(x)|p_n(x)|^q\,dx.$$

Relation (8) of Theorem 3 yields an L_q Marcinkiewicz—Zygmund type result

$$\frac{1}{2}\sum_{j=0}^{mn-1}(x_{j+1}-x_{j-1})w(x_j)|t_n(x_j)|^q = \bigl(1+O(m^{-2})\bigr)\int_{-\pi}^{\pi} w(x)|t_n(x)|^q\,dx \quad \forall\ t_n \in T_n$$

when $\max_{0\le j\le mn-1}(x_{j+1}-x_j) \sim \dfrac{1}{nm},\ 2 \le q < \infty$. This raises the natural question if the quadratic accuracy provided by the term $O(m^{-2})$ in the above relation is the best possible. It turns out that in general the term $O(m^{-2})$ cannot be improved further. This is shown by the next theorem in the model case of L_2 norm.

Theorem 4. *There exist points $-\pi = x_0 < x_1 < \cdots < x_{mn} = \pi$ with*

$$\max_{0\le j\le mn-1}(x_{j+1}-x_j) = \frac{\pi}{4mn}$$

such that

$$\frac{1}{2}\sum_{j=0}^{mn-1}(x_{j+1}-x_{j-1})\cos^2 nx_j \le \Bigl(1-\frac{\pi^2}{8^3m^2}\Bigr)\int_{-\pi}^{\pi}\cos^2 nx\,dx.$$

P r o o f. Set $h = h_{mn} = \dfrac{\pi}{8mn},\ z_j := \dfrac{\pi}{4n}-2hj,\ 0 \le j \le m-1$, and $y_j := \dfrac{\pi}{4n}+hj,\ 1 \le j \le 2m$. This provides a total of $3m$ distinct points on the interval $\Bigl[0,\dfrac{\pi}{2n}\Bigr]$ which are $2h$-equidistant on $\Bigl[0,\dfrac{\pi}{4n}\Bigr]$ and h-equidistant on $\Bigl[\dfrac{\pi}{4n},\dfrac{\pi}{2n}\Bigr]$. We extend this system of points to $[-\pi,\pi]$ by symmetry about the origin and $\dfrac{\pi}{n}$ periodicity. This yields a total of $12mn$ distinct points on $[-\pi,\pi]$ which are denoted by $-\pi = x_0 < x_1 < \cdots < x_N = \pi,\ N := 12mn-1$.

Then similarly to (10) we obtain this estimate with $f_n(x) = \cos^2 nx,\ f_n''(x) = -2n^2 \cos 2nx$

$$R := \int_{-\pi}^{\pi} \cos^2 nx\, dx - \frac{1}{2}\sum_{j=0}^{N-1}(x_{j+1} - x_{j-1})\cos^2 nx_j = n^2 \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} (x - x_j)(x_{j+1} - x)\cos 2nx\, dx.$$

Hence by the $\frac{\pi}{n}$ periodicity and central symmetry of the point system

$$R = 4n^3 \sum_{[x_j, x_{j+1}] \subset [0, \frac{\pi}{2n})} \int_{x_j}^{x_{j+1}} (x - x_j)(x_{j+1} - x)\cos 2nx\, dx.$$

Furthermore,

$$\sum_{[x_j, x_{j+1}] \subset [0, \frac{\pi}{2n})} \int_{x_j}^{x_{j+1}} (x - x_j)(x_{j+1} - x)\cos 2nx\, dx = \sum_{0 \le j \le m-1} \Bigg(\int_{z_{j+1}}^{z_j} (x - z_{j+1})(z_j - x)\cos 2nx\, dx$$

$$+ \int_{y_{2j}}^{y_{2j+1}} (y - y_{2j})(y_{2j+1} - y)\cos 2ny\, dy + \int_{y_{2j+1}}^{y_{2j+2}} (y - y_{2j+1})(y_{2j+2} - y)\cos 2ny\, dy \Bigg).$$

Moreover, substituting $y = \frac{\pi}{2n} - x$ in the last two integrals and noting that $y_{2j} = \frac{\pi}{2n} - z_j,\ y_{2j+1} = \frac{\pi}{2n} - z_j + h$, we see that

$$\int_{z_{j+1}}^{z_j} (x - z_{j+1})(z_j - x)\cos 2nx\, dx + \int_{y_{2j}}^{y_{2j+1}} (y - y_{2j})(y_{2j+1} - y)\cos 2ny\, dy$$

$$+ \int_{y_{2j+1}}^{y_{2j+2}} (y - y_{2j+1})(y_{2j+2} - y)\cos 2ny\, dy = h \int_{z_{j+1}}^{z_j} (h - |x - z_j + h|)_+ \cos 2nx\, dx.$$

Combining the last three relations we obtain

$$R = 4hn^3 \sum_{0 \le j \le m-1} \int_{z_{j+1}}^{z_j} (h - |x - z_j + h|)_+ \cos 2nx\, dx \ge 4hn^3 \sum_{m/2 \le j \le m-1} \int_{z_{j+1}}^{z_j} (h - |x - z_j + h|)_+ \cos 2nx\, dx$$

$$\ge \sqrt{8} hn^3 \sum_{m/2 \le j \le m-1} \int_{z_{j+1}}^{z_j} (h - |x - z_j + h|)_+\, dx$$

$$= \sqrt{8} hn^3 \sum_{m/2 \le j \le m-1} h^2 \ge h^3 n^3 m = \frac{\pi^2}{8^3 m^2} \int_{-\pi}^{\pi} \cos^2 nx\, dx.$$

This estimate immediately implies the required lower bound. □

R e m a r k. Theorems 2 and 3 together with Corollaries 1 and 2 provide an L_q Marcinkiewicz—Zygmund type result for univariate algebraic and trigonometric polynomials with a remainder term of quadratic accuracy in the case when $2 \le q < \infty$. Moreover, by Theorem 4 this quadratic error term is in general sharp when $q = 2$. The sharpness in the case $q > 2$ could be verified similarly. However, for $1 \le q < 2$ our method yields only a weaker $O(m^{-1})$ error term instead of $O(m^{-2})$. The question of determining the sharp error term for $1 \le q < 2$ appears to be an interesting open problem.

3. Refined Marcinkiewicz—Zygmund inequalities for multivariate polynomials

Now we extend our considerations to the multivariate case. The main building block in several variables consists in verifying a Marcinkiewicz—Zygmund type result with a quadratic error term for the 2-dimensional unit disc $B^2 \subset \mathbb{R}^2$. Then using the technique developed in [4] one can apply certain transformations of the domain like rotation or symmetry to obtain a similar result for the d-dimensional ball and simplex.

Theorem 5. *Consider the weight $w^*(x,y) := w(r)\phi(t),\ x = r\cos t,\ y = r\sin t$, where $w(r)$ and $\phi(t)$ are Jacobi type weights* (13), *$w(r)$ is even on $[-1,1]$, and $\phi(t) = \phi(t+\pi),\ t \in \mathbb{R}$. Then setting $t_j := \dfrac{\pi j}{mn},\ 0 \le j \le 2mn$, we have for every $q \ge 2$ and $p_n \in P_n^2$*

$$(1 + O(m^{-2})) \int\limits_{B^2} |p_n(x,y)|^q w^*(x,y)\, dxdy = \Big(\frac{\pi}{mn}\Big)^2 \sum_{k,j=0}^{mn-1} a_{j,k} |p_n(\cos t_k e^{it_j})|^q,$$

where $a_{k,j} := \phi(t_j) w(\cos t_k) |\sin 2t_k|/2,\ 0 \le k, j \le 2mn$, and $O(m^{-2})$ depends only on q and w^*.

P r o o f. Using the polar coordinates $x = r\cos t,\ y = r\sin t$ and the relation $\phi(t) = \phi(t+\pi)$ we have

$$\|p_n\|_{L_q(w^*)}^q = \int\limits_{B^2} |p_n|^q w^*\, dxdy$$

$$= \frac{1}{2}\int\limits_0^{2\pi} \phi(t) \int\limits_{-1}^{1} |p_n(r\cos t, r\sin t)|^q w(r)|r|\, drdt = \frac{1}{2}\int\limits_0^{2\pi} \phi(t) g(t)\, dt, \quad p_n \in P_n^2,$$

where

$$g(t) := \int\limits_{-1}^{1} |p_n(r\cos t, r\sin t)|^q w(r)|r|\, dr.$$

Then similarly to (10) we obtain

$$R := \left|\frac{1}{2}\int\limits_0^{2\pi} \phi(t)g(t)\, dt - \frac{\pi}{2mn}\sum_{j=0}^{2mn-1} \phi(t_j)g(t_j)\right| \le \Big(\frac{\pi}{2mn}\Big)^2 \int\limits_0^{2\pi} |(\phi g)''|\, dt$$

$$\le \Big(\frac{\pi}{2mn}\Big)^2 \Bigg(\int\limits_0^{2\pi} |\phi'' g|\, dt + 2\int\limits_0^{2\pi} |\phi' g'|\, dt + \int\limits_0^{2\pi} |\phi g''|\, dt\Bigg). \tag{18}$$

Similarly to the proof of Theorem 3 the three terms on the right-hand side of (18) can be estimated using Hölder, Schur, and Bernstein type inequalities together with the Fubini theorem. The Schur type inequality (14) will be used below with the Jacobi weights

$$w_1(t) := \prod_{1\le k\le s} |t - y_k|, \quad d_{w_1} = 1, \quad w_2(t) := \prod_{1\le k\le s} |t - y_k|^2, \quad d_{w_2} = 2,$$

where $\phi(t) = \prod_{1\le k\le s} |t - y_k|^{a_k},\ a_k \ge 1$. It can be easily verified that with some $c > 0$ depending only on the weight

$$|\phi'(t)| w_1(t) \le c\phi(t), \quad |\phi''(t)| w_2(t) \le c\phi(t), \quad t \in [0, 2\pi]. \tag{19}$$

1st term. Then by the Fubini theorem and L_q Schur type inequality (14) applied to the trigonometric polynomial $p_n(r\cos t, r\sin t)$ of the variable t with the doubling weight $|\phi''(t)|$ and $w_2(t)$ we obtain by (19)

$$\int_0^{2\pi} |\phi''| g\, dt = \int_{[-1,1]} w(r)|r| \int_0^{2\pi} |\phi''(t)||p_n(r\cos t, r\sin t)|^q\, dtdr$$

$$\leq cn^2 \int_{[-1,1]} w(r)|r| \int_0^{2\pi} |\phi''(t)| w_2(t) |p_n(r\cos t, r\sin t)|^q\, dtdr$$

$$\leq cn^2 \int_{[-1,1]} w(r)|r| \int_0^{2\pi} \phi(t)|p_n(r\cos t, r\sin t)|^q\, dtdr = cn^2 \int_{B^2} |p_n|^q w^*\, dxdy.$$

2nd term. First we use the Hölder inequality to obtain for any $r \in [0,1]$ and $p_n = p_n(r\cos t, r\sin t)$

$$\int_{[0,2\pi]} |p_n|^{q-1} \Big|\frac{\partial p_n}{\partial t}\Big| |\phi'(t)|\, dt \leq \bigg(\int_{[0,2\pi]} \Big|\frac{\partial p_n}{\partial t}\Big|^q |\phi'(t)|\, dt\bigg)^{1/q} \bigg(\int_{[0,2\pi]} |p_n|^q |\phi'|(t)\, dt\bigg)^{(q-1)/q}.$$

Since for every $r \in [-1,1]$, $p_n = p_n(r\cos t, r\sin t)$ is a univariate trigonometric polynomial of degree at most n we have by the L_q Bernstein inequality (12) with the doubling weight $|\phi'(t)|$

$$\bigg(\int_{[0,2\pi]} \Big|\frac{\partial p_n}{\partial t}\Big|^q |\phi'(t)|\, dt\bigg)^{1/q} \leq cn \bigg(\int_{[0,2\pi]} |p_n|^q |\phi'(t)|\, dt\bigg)^{1/q} \quad \forall\ r \in [-1,1].$$

Combining the last two estimates yields

$$\int_{[0,2\pi]} |p_n|^{q-1} \Big|\frac{\partial p_n}{\partial t}\Big| |\phi'(t)|\, dt \leq cn \int_{[0,2\pi]} |p_n|^q |\phi'(t)|\, dt \quad \forall\ r \in [-1,1].$$

Hence using the above estimate and the Fubini theorem, we get

$$\int_{[0,2\pi]} |g'(t)\phi'(t)|\, dt \leq q \int_{[-1,1]} w(r)|r| \int_{[0,2\pi]} |p_n|^{q-1} \Big|\frac{\partial p_n}{\partial t}\Big| |\phi'(t)|\, dtdr$$

$$\leq cqn \int_{[-1,1]} w(r)|r| \int_{[0,2\pi]} |p_n|^q |\phi'(t)|\, dtdr = cqn \int_{[0,2\pi]} g(t)|\phi'(t)|\, dt. \tag{20}$$

Now by the L_q Schur type inequality (14) applied to the trigonometric polynomial $p_n(r\cos t, r\sin t)$ of variable t with the doubling weight $|\phi'(t)|$ and $w_1(t)$ we obtain by (19)

$$\int_{[0,2\pi]} g(t)|\phi'(t)|\, dt = \int_{[-1,1]} w(r)|r| \int_{[0,2\pi]} |\phi'(t)||p_n(r\cos t, r\sin t)|^q\, dtdr$$

$$\leq cn \int_{[-1,1]} w(r)|r| \int_{[0,2\pi]} \phi(t)|p_n(r\cos t, r\sin t)|^q\, dtdr = cn \int_{B^2} |p_n|^q w^*\, dxdy.$$

This estimate together with (20) yields

$$\int_{[0,2\pi]} |g'(t)\phi'(t)|\, dt \leq cn^2 \int_{B^2} |p_n|^q w^*\, dxdy.$$

3rd term. Again by the Hölder and L_q Bernstein type inequality (12) with the doubling weight ϕ applied for the first and second derivatives of the trigonometric polynomial $p_n(r\cos t, r\sin t)$ of the variable t

$$\int\limits_{[0,2\pi]} |p_n|^{q-1}\Big|\frac{\partial^2 p_n}{\partial t^2}\Big|\phi(t)\,dt + \int\limits_{[0,2\pi]} |p_n|^{q-2}\Big(\frac{\partial p_n}{\partial t}\Big)^2\phi(t)\,dt$$

$$\leq \Big(\int\limits_{[0,2\pi]}\Big|\frac{\partial^2 p_n}{\partial t^2}\Big|^q\phi\,dt\Big)^{1/q}\Big(\int\limits_{[0,2\pi]}|p_n|^q\phi\,dt\Big)^{(q-1)/q} + \Big(\int\limits_{[0,2\pi]}\Big|\frac{\partial p_n}{\partial t}\Big|^q\phi\,dt\Big)^{2/q}\Big(\int\limits_{[0,2\pi]}|p_n|^q\phi\,dt\Big)^{(q-2)/q}$$

$$\leq cn^2\int\limits_{[0,2\pi]}|p_n|^q\phi\,dt.$$

Thus

$$\int\limits_0^{2\pi}|\phi g''|\,dt \leq q^2\int\limits_{[-1,1]} w(r)|r|\int\limits_{[0,2\pi]}\Big(|p_n|^{q-1}\Big|\frac{\partial^2 p_n}{\partial t^2}\Big| + |p_n|^{q-2}\Big(\frac{\partial p_n}{\partial t}\Big)^2\Big)\phi\,dtdr$$

$$\leq cn^2q^2\int\limits_{[-1,1]} w(r)|r|\int\limits_{[0,2\pi]}|p_n|^q\phi\,dtdr = cn^2\int\limits_{B^2}|p_n|^q w^*\,dxdy.$$

Summarizing we can see that each integral on the right-hand side of (18) can be estimated by $cn^2\int_{B^2}|p_n|^q w^*\,dxdy$ yielding

$$\frac{\pi}{2mn}\sum_{j=0}^{2mn-1}\phi(t_j)g(t_j) = \frac{\pi}{mn}\sum_{j=0}^{mn-1}\phi(t_j)g(t_j) = \big(1+O(m^{-2})\big)\int\limits_{B^2}|p_n|^q w^*\,dxdy. \tag{21}$$

Furthermore,

$$g(t_j) := \int\limits_{[-1,1]}|p_n(r\cos t_j, r\sin t_j)|^q w(r)|r|\,dr,$$

where each $p_n(r\cos t_j, r\sin t_j)$, $0\leq j\leq 2mn$, is an algebraic polynomial of degree at most n of the variable $r\in[-1,1]$. Therefore Corollary 2 is applicable now to every polynomial $p_n(r\cos t_j, r\sin t_j)$, $1\leq j\leq mn$, with the doubling weight $w(r)|r|$ and nodes $r_k := \cos t_k$ yielding

$$\big(1+O(m^{-2})\big)\frac{\pi}{mn}\sum_{k=0}^{mn-1}\sqrt{1-r_k^2}|r_k|w(r_k)|p_n(r_ke^{it_j})|^q = \int\limits_{[-1,1]}|p_n(r\cos t_j, r\sin t_j)|^q w(r)|r|\,dr = g(t_j).$$

Using this relation combined with (21) we obtain

$$\big(1+O(m^{-2})\big)\int\limits_{B^2}|p_n|^q w^*\,dxdy = \Big(\frac{\pi}{mn}\Big)^2\sum_{k,j=0}^{mn-1}a_{k,j}\big|p_n(r_ke^{it_j})\big|^q,$$

where

$$a_{k,j} := \phi(t_j)w(r_k)|r_k|\sqrt{1-r_k^2} = \frac{\phi(t_j)w(\cos t_k)|\sin 2t_k|}{2}, \quad 0\leq k,j\leq mn. \qquad \square$$

Theorem 5 can be used to obtain similar Marcinkiewicz—Zygmund type result with a quadratic error term for various other multivariate domains, like for a instance simplex or a ball. If for example $\Delta := \{(u,v)\in\mathbb{R}:\ 0\leq u,v\leq 1,\ u+v\leq 1\}$ is the standard simplex on the plane then evidently

$$\int\limits_{\Delta}|p_n(u,v)|^q\,dudv = \int\limits_{B^2}|p_n(x^2,y^2)|^q|xy|\,dxdy$$

and hence Theorem 5 is immediately applicable with $w^*(x,y) := |xy|,\ w(r) = r^2,\ \phi(t) = \sin t \cos t$.

Furthermore, if we consider the unit ball $B^3 \subset \mathbb{R}^3$ then clearly using the cylindrical coordinates

$$\int\limits_{B^3} |p_n|^q = \frac{1}{2} \int\limits_{[0,2\pi]} \int\limits_{B^2} |p_n(x, y\sin t, y\cos t)|^q |y|\, dxdydt.$$

Now Theorem 5 is applicable for the integral on B^2 with $w^*(x,y) := |y|$ and bivariate algebraic polynomial $p_n(x, y\sin t, y\cos t) = g_n(x,y)$ of the variables x, y. Subsequently, we can also use Theorem 3 for the univariate trigonometric polynomials $p_n(x_j, y_k \sin t, y_k \cos t)$ of variable t in order to obtain proper discrete expressions for the integrals $\int_{[0,2\pi]} |p_n(x_j, y_k \sin t, y_k \cos t)|^q\, dt$. This will yield a Marcinkiewicz—Zygmund type result with a *quadratic error term* for the ball $B^3 \subset \mathbb{R}^3$. We refer the reader to [4] for the discussion of various geometric transformations which enable to pass to new multivariate domains in Marcinkiewicz—Zygmund type inequalities. This can lead to deriving additional Marcinkiewicz—Zygmund type results with a quadratic error term.

Let us also mention that the technique used in the proof of Theorem 5 allows to extend the Marcinkiewicz—Zygmund type result given for univariate trigonometric polynomials in Corollary 1 to the case of multivariate trigonometric polynomials. Let us denote by T_n^d the space of real trigonometric polynomials of d variables and degree at most n in each variable.

Theorem 6. *For any* $2 \le q < \infty,\ n \in \mathbb{N},\ d \ge 1,$ *and every* $t_n \in T_n^d$ *we have*

$$\Big(\frac{\pi}{mn}\Big)^d \sum_{\boldsymbol{j} \in \mathbb{Z}_{mn}^d} \Big| t_n\Big(\frac{\pi \boldsymbol{j}}{mn}\Big)\Big|^q = \Big(1 + O\Big(\frac{1}{m^2}\Big)\Big) \int\limits_{[-\pi,\pi]^d} |t_n(\boldsymbol{x})|^q\, d\boldsymbol{x},$$

where $\mathbb{Z}_{mn}^d := \{\boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d :\ -mn \le k_j \le mn - 1,\ 1 \le j \le d\}$.

P r o o f. We outline the proof by induction on the dimension d. For $d = 1$ the statement of the theorem is given by Corollary 1. Assume that Theorem 6 holds for $d-1,\ d \ge 2$. Clearly,

$$\int\limits_{[-\pi,\pi]^d} |t_n(\mathbf{x})|^q\, d\mathbf{x} = \int\limits_{[-\pi,\pi]} \int\limits_{[-\pi,\pi]^{d-1}} |t_n(\mathbf{y}, t)|^q\, d\mathbf{y}dt = \int\limits_{[-\pi,\pi]} g(t)\, dt,$$

where

$$g(t) := \int\limits_{[-\pi,\pi]^{d-1}} |t_n(\mathbf{y}, t)|^q\, d\mathbf{y}.$$

Then similarly to (10)

$$\Bigg| \int\limits_{[-\pi,\pi]} g(t)\, dt - \frac{\pi}{mn} \sum_{-mn \le j \le mn-1} \Big| g\Big(\frac{\pi j}{mn}\Big)\Big| \Bigg| \le \frac{1}{2}\Big(\frac{\pi}{mn}\Big)^2 \int_{-\pi}^{\pi} |g''(t)|\, dt.$$

Furthermore, analogously to (11) and the subsequent estimates in the proof of Theorem 2 it can be shown that

$$\int_{-\pi}^{\pi} |g''(t)|\, dt \le q^2 n^2 \int\limits_{[-\pi,\pi]^d} |t_n(\mathbf{x})|^q\, d\mathbf{x}.$$

Thus combining the above relations yields

$$\Big(1 + O\Big(\frac{1}{m^2}\Big)\Big) \int\limits_{[-\pi,\pi]^d} |t_n(\mathbf{x})|^q\, d\mathbf{x} = \frac{\pi}{mn} \sum_{-mn \le j \le mn-1} \Big| g\Big(\frac{\pi j}{mn}\Big)\Big|.$$

Now it remains to note that

$$g\Big(\frac{\pi j}{mn}\Big) = \int\limits_{[-\pi,\pi]^{d-1}} \Big|t_n\Big(\mathbf{y}, \frac{\pi j}{mn}\Big)\Big|^q \, d\mathbf{y},$$

where $t_n\Big(\mathbf{y}, \dfrac{\pi j}{mn}\Big) \in T_n^{d-1},\ 1 \le j \le d$. Hence the proof can be completed by using the induction hypothesis for every $\displaystyle\int_{[-\pi,\pi]^{d-1}} \Big|t_n\Big(\mathbf{y}, \frac{\pi j}{mn}\Big)\Big|^q d\mathbf{y},\ 1 \le j \le d.$ □

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