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STECHKIN'S PROBLEM ON THE BEST APPROXIMATION OF AN UNBOUNDED OPERATOR BY BOUNDED ONES AND RELATED PROBLEMS

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This paper discusses Stechkin's problem on the best approximation of a linear unbounded operator by bounded linear operators and related extremal problems. The main attention is paid to the approximation of differentiation operators in Lebesgue spaces on the axis and to the operator of the continuation of an analytic function to a domain from a part of the boundary of the domain. This is a review paper based on the materials of the authors' lecture on September 14, 2020, at the X Internet video-conference "Day of Mathematics and Mechanics" of four institutes of the Russian Academy of Sciences: Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of RAS (Yekaterinburg), Sobolev Institute of the Siberian Branch of RAS (Novosibirsk), Steklov Mathematical Institute (Moscow), and the St. Petersburg Department of the Steklov Mathematical Institute. The lecture of the authors was dedicated to the 100th anniversary of the birth of Sergei Borisovich Stechkin. The problem of the best approximation of a linear unbounded operator by bounded ones is one of his legacies. We tried to at least partially reflect the new results, methods, and statements that appeared in this topic after the publication of the review papers (Arestov, Gabushin, 1995–1996). The material on this topic is wide; the selection of the material for the lecture and paper is the responsibility of the authors.

Keywords: Stechkin's problem, recovery, unbounded linear operator, differentiation operator, Kolmogorov inequality, analytic functions, boundary values.

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