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ON THE NORMS OF BOMAN–SHAPIRO DIFFERENCE OPERATORS

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For given $k \in \mathbb{N}$ and $h > 0$, an exact inequality $\|W_{2k}(f,h)\|_C \le C_k \|f\|_C$ is considered on the space $C = C(\mathbb{R})$ of continuous functions bounded on the real axis $\mathbb{R} = (-\infty, \infty)$ for the Boman–Shapiro difference operator $W_{2k}(f,h)(x) := \dfrac{(-1)^k}{h}\displaystyle\int_{-h}^{h}\binom{2k}{k}^{-1}\widehat{\Delta}_t^{2k} f(x)\Big(1-\frac{|t|}{h}\Big)\,dt$, where $\widehat{\Delta}_t^{2k} f(x) := \sum_{j=0}^{2k}(-1)^j\binom{2k}{j} f(x+jt-kt)$ is the central finite difference of a function f of order $2k$ with step t. For each fixed $k \in \mathbb{N}$, the exact constant C_k in the above inequality is the norm of the operator $W_{2k}(\cdot,h)$ from C to C. It is proved that C_k is independent of h and increases in k. A simple method is proposed for the calculation of the constant $C_* = \lim_{k\to\infty} C_k = 2.6699263\ldots$ with accuracy 10^{-7}. We also consider the problem of extending a continuous function f from the interval $[-1,1]$ to the axis $\mathbb{R}$. For extensions $g_f := g_{f,k,h}$, $k \in \mathbb{N}$, $0 < h < 1/(2k)$, of functions $f \in C[-1,1]$, we obtain new two-sided estimates for the exact constant C_k^* in the inequality $\|W_{2k}(g_f,h)\|_{C(\mathbb{R})} \le C_k^*\,\omega_{2k}(f,h)$, where $\omega_{2k}(f,h)$ is the modulus of continuity of f of order $2k$. Specifically, for any natural $k \ge 6$ and any $h \in \big(0, 1/(2k)\big)$, we prove the double inequality $5/12 \le C_k^* < \big(2+e^{-2}\big)\,C_*$.

Keywords: difference operator, kth modulus of continuity, norm estimate.

MSC: 41A10, 41A17, 41A44

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