

ANALOG OF THE HADAMARD THEOREM AND RELATED EXTREMAL PROBLEMS ON THE CLASS OF ANALYTIC FUNCTIONS

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We study several related extremal problems for analytic functions in a finitely connected domain G with rectifiable Jordan boundary Γ . A sharp inequality is established between values of a function analytic in G and weighted means of its boundary values on two measurable subsets γ_1 and $\gamma_0 = \Gamma \setminus \gamma_1$ of the boundary:

$$|f(z_0)| \leq C \|f\|_{L^q_{\varphi_1}(\gamma_1)}^\alpha \|f\|_{L^p_{\varphi_0}(\gamma_0)}^\beta, \quad z_0 \in G, \quad 0 < q, p \leq \infty.$$

The inequality is an analog of Hadamard's three-circle theorem and the Nevanlinna brothers' theorem on two constants. In the case of a doubly connected domain G and $1 \leq q, p \leq \infty$, we study the cases where the inequality provides the value of the modulus of continuity for a functional of analytic extension of a function from a part of γ_1 to a given point of the domain. In these cases, the corresponding problems of optimal recovery of a function from its approximate boundary values on γ_1 and of the best approximation of a functional by linear bounded functionals are solved. The case of a simply connected domain G has been completely investigated previously.

Keywords: analytic functions, optimal recovery of a functional, best approximation of an unbounded functional by bounded functionals, harmonic measure.

MSC: 30C85, 65E05, 30H99

DOI: 10.21538/0134-4889-2020-26-4-32-4

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Received July 13, 2020

Revised October 5, 2020

Accepted October 26, 2020

Funding Agency: This work was supported by the Russian Foundation for Basic Research (project no. 18-01-00336) and by the Russian Academic Excellence Project (agreement no. 02.A03.21.0006 of August 27, 2013, between the Ministry of Education and Science of the Russian Federation and Ural Federal University), and as part of research conducted in the Ural Mathematical Center.

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Cite this article as: R. R. Akopyan. Analog of the Hadamard theorem and related extremal problems on the class of analytic functions, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2020, vol. 26, no. 4, pp. 32–47.