TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN

Vol. 26 No. 4 **2020**

ON THE CONNECTION OF SOME GROUPS GENERATED BY 3-TRANSPOSITIONS WITH COXETER GROUPS

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Coxeter groups, more commonly known as reflection-generated groups, have numerous applications in various fields of mathematics and beyond. Groups with Fischer's 3-transpositions are also related to many structures: finite simple groups, triple graphs, geometries of various spaces, Lie algebras, etc. The intersection of these classes of groups consists of finite Weyl groups $W(A_n) \simeq S_{n+1}$, $W(D_n)$, and $W(E_n)$ $(n = 6, 7, 8)$ of simple finite-dimensional algebras and Lie groups. The paper continues the study of the connection between the finite groups $Sp_{2l}(2)$ and $O^{\pm}_{2l}(2)$ from clauses (ii)–(iii) of Fischer's theorem and infinite Coxeter groups. The organizing basis of the connection under study is general Coxeter tree graphs Γ_n with vertices $1, \ldots, n$. To each vertex i of the graph Γ_n, we assign the generating involution (reflection) s_i of the Coxeter group G_n, the basis vector e_i of the space V_n over the field F_2 of two elements, and the generating transvection w_i of the subgroup $W_n = \langle w_1, \ldots, w_n \rangle$ of $SL(V_n) = SL_n(2)$. The graph Γ_n corresponds to exactly one Coxeter group of rank n: $G_n = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}}, m_{ij} \le 3 \rangle$, where $m_{ii} = 1$, $1 \le i < j \le n$, and $m_{ij} = 3$ or $m_{ij} = 2$ depending on whether Γ_n contains the edge (i, j). The form defined by the graph Γ_n turns V_n into an orthogonal space whose isometry group W_n is generated by the mentioned transvections (3-transpositions) $w_1, \ldots, w_n$; in this case, the relations $(w_i w_j)^{m_{ij}} = 1$ hold in W_n and, therefore, the mapping $s_i \to w_i$ $(i = 1, \ldots, n)$ is continued to the surjective homomorphism $G_n \to W_n$. In the authors' previous paper, for all groups $W_n = O^{\pm}_{2l}(2)$ $(n = 2l \ge 6)$ and $W_n = Sp_{2l}(2)$ $(n = 2l + 1 \ge 7)$, an algorithm was given for enumerating the corresponding tree graphs Γ_n by grouping them according to E-series of nested graphs. In the present paper, a close genetic connection is established between the groups $O^{\pm}_{2l}(2)$ and $Sp_{2l}(2) \times \mathbb{Z}_2$ $(3 \le l \le 10)$ and the corresponding (infinite) Coxeter groups G_n with the difference in their genetic codes by exactly one gene (relation). For the groups W_n with the graphs Γ_n from the E-series $\{E_n\}$, $\{I_n\}$, $\{J_n\}$, and $\{K_n\}$, additional word relations are written explicitly.

Keywords: groups with 3-transpositions, Coxeter graphs and groups, genetic codes.

MSC: 20C40

DOI: 10.21538/0134-4889-2020-26-4-234-243

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Received May 19, 2020
Revised November 4, 2020
Accepted November 16, 2020

Funding Agency: This work was supported by the Russian Foundation for Basic Research (project no. 19-01-00566 A.)

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Cite this article as: V. M. Sinitsin, A. I. Sozutov. On the connection of some groups generated by 3-transpositions with Coxeter groups, *Trudy Instituta Matematiki i Mekhaniki UrO RAN*, 2020, vol. 26, no. 4, pp. 234–243 .