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SOME SUPPLEMENTS TO S. B. STECHKIN'S INEQUALITIES IN DIRECT AND INVERSE THEOREMS ON THE APPROXIMATION OF CONTINUOUS PERIODIC FUNCTIONS

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We give some supplements and comments to inequalities between elements of the sequence of best approximations $\{E_{n-1}(f)\}_{n=1}^{\infty}$ and the kth-order moduli of smoothness $\omega_k(f^{(r)};\delta)$, $\delta\in[0,+\infty)$, of a function $f\in C^r(\mathbb{T})$, where $k\in\mathbb{N}$, $r\in\mathbb{Z}_+$, $f^{(0)}\equiv f$, $C^0(\mathbb{T})\equiv C(\mathbb{T})$, and $\mathbb{T}=(-\pi,\pi]$, which were published by S. B. Stechkin in 1951 in the study of direct and inverse theorems of approximation of 2π-periodic continuous functions. In particular, we prove the following results:

(a) the direct theorem or the Jackson–Stechkin inequality: $E_{n-1}(f)\le C_1(k)\omega_k(f;\pi/n)$, $n\in\mathbb{N}$, can be strengthened as $E_{n-1}(f)\le \rho_n^{(k)}(f)\equiv n^{-k}\max\{\nu^kE_{\nu-1}(f)\colon 1\le\nu\le n\}\le 2^kC_1(k)\omega_k(f;\pi/n)$, $n\in\mathbb{N}$. This inequality is order-sharp on the class of all functions $f\in C(\mathbb{T})$ with a given majorant or with a given decrease order of the modulus of smoothness $\omega_k(f;\delta)$; namely: for any $k\in\mathbb{N}$ and $\omega\in\Omega_k(0,\pi]$, there exists a function $f_0(\,\cdot\,;\omega)\in C(\mathbb{T})$ (f_0 is even for odd k and is odd for even k) such that $\omega_k(f_0;\delta)\asymp C_2(k)\omega(\delta)$, $\delta\in(0,\pi]$. Moreover, order equalities hold: $E_{n-1}(f_0)\asymp C_3(k)\rho_n^{(k)}(f_0)\asymp C_4(k)\omega_k(f_0;\pi/n)\asymp C_5(k)\omega(\pi/n)$, $n\in\mathbb{N}$, where $\Omega_k(0,\pi]$ is the class of functions $\omega=\omega(\delta)$ defined on $(0,\pi]$ and such that $0<\omega(\delta)\downarrow 0$ $(\delta\downarrow 0)$ and $\delta^{-k}\omega(\delta)\downarrow$ $(\delta\uparrow)$;

(b) a necessary and sufficient condition under which the inverse theorem (without the derivatives), or the Salem–Stechkin inequality $\omega_k(f;\pi/n)\le C_6(k)n^{-k}\sum_{\nu=1}^{n}\nu^{k-1}E_{\nu-1}(f)$, $n\in\mathbb{N}$, holds is Stechkin's inequality $\|T_n^{(k)}(f)\|\le C_7(k)\sum_{\nu=1}^{n}\nu^{k-1}E_{\nu-1}(f)$, $n\in\mathbb{N}$, where $T_n(f)\equiv T_n(f;x)$ is a trigonometric polynomial of best $C(\mathbb{T})$-approximation to the function f (i.e., $\|f-T_n(f)\|=E_n(f)$, $n\in\mathbb{Z}_+$);

(c) the inverse theorem (with the derivatives), or the Vallée-Poussin–Stechkin inequality $\omega_k(f^{(r)};\pi/n)\le C_8(k,r)\big\{n^{-k}\sum_{\nu=1}^{n}\nu^{k+r-1}E_{\nu-1}(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_{\nu-1}(f)\big\}$ for any $n\in\mathbb{N}$, as well as Stechkin's earlier inequality $E_{n-1}(f^{(r)})\le C_9(r)\big\{n^rE_{n-1}(f)+\sum_{\nu=n+1}^{\infty}\nu^{r-1}E_{\nu-1}(f)\big\}$, $n\in\mathbb{N}$, where $E(f;r)\equiv\sum_{n=1}^{\infty}n^{r-1}E_{n-1}(f)<\infty$ (by S. N. Bernstein's theorem, this inequality guarantees that f lies in $C^r(\mathbb{T})$, where $r\in\mathbb{N}$) can be supplemented with the following key inequalities: $\|f^{(r)}\|\le C_{10}(r)E(f;r)$ and $\|T_n^{(r)}(f)\|\le C_7(r)\sum_{\nu=1}^{n}\nu^{r-1}E_{\nu-1}(f)$, $n\in\mathbb{N}$. Moreover, all the inequalities formulated in this paragraph are pairwise equivalent; i.e., any of these inequalities implies any other and, hence, all the inequalities.

Keywords: best approximation, modulus of smoothness, direct theorem, inverse theorem, order equality, equivalent inequalities, order-sharp inequality on a class.

MSC: 42A10, 41A17, 41A25, 42A32

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