

## Chebyshev's Problem on Extremal Values of Moments of Nonnegative Algebraic Polynomials

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The paper is devoted to the study of the 1883 Chebyshev problem on the maximum and minimum values of the moments of nonnegative algebraic polynomials with a fixed zero moment on finite and infinite intervals with weight. For the first moment on the segment  $[-1, 1]$ , the problem was solved by P.L. Chebyshev in the case of unit weight and by G. Szegő in the general case. The extreme values of the first moment were the largest and smallest zeros of some orthogonal polynomials. Their solution used the representation of non-negative polynomials and the Gauss quadrature formula of the highest accuracy. We solve the Chebyshev problem on the extreme values of the moments of order  $k \geq 2$  for any intervals  $(a, b)$  if  $k$  is odd and for intervals with  $a \geq 0$  or  $b \leq 0$  if  $k$  is even. These intervals are characterized by the fact that the function  $x^k$  is monotone on them. As for  $k = 1$ , the extremal values of moments of order  $k$  are the  $k$ th powers of the largest and least zeros of some orthogonal polynomials. The other zeros of these polynomials are also extreme. They give extreme values in a generalization of the Chebyshev problem to the case of polynomials with fixed zeros. To solve the generalized Chebyshev problem, we constructed special quadrature formulae. The solution to the Chebyshev problem is obtained as a special case of the solution to the generalized Chebyshev problem. In some cases, due to the absence of the second endpoint for an infinite interval, it is not possible to construct the necessary quadrature formulae and one has to directly solve the Chebyshev problem, relying on the representation of non-negative polynomials. In addition to the cases noted above, for moments of even order, it is possible to solve the Chebyshev problem on an interval  $(-a, a)$  if the weight is even. In the general case, the question of solving the Chebyshev problem for moments of even order remains open.

Keywords: moments of algebraic polynomials, Chebyshev's problem, quadrature formulae, orthogonal polynomials.

MSC: 33C45, 41A17, 41A55

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