BOUNDS OF THE NIKOL'SKII POLYNOMIAL CONSTANTS IN L^P WITH GEGENBAUER WEIGHT

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We study bounds and the asymptotics behavior as $n \to \infty$ of the sharp Nikol'skii constant in the inequality $\|u\|_\infty \leq \mathcal{C}_\alpha(n)\|u\|_p$ for trigonometric and algebraic polynomials of degree at most n in the space L^p on $(-\pi, \pi)$ with periodic Gegenbauer weight $|\sin x|^{2\alpha+1}$ and on $[-1, 1]$ with algebraic Gegenbauer weight $(1-x^2)^\alpha$, respectively. We prove that for $p \geq 1$ and all $\alpha \geq -1/2$ we have $\mathcal{C}_\alpha(n) \sim \mathcal{L}_p n^{(2\alpha+2)/p}$, where $\mathcal{L}_p$ is the sharp Nikol'skii constant for entire functions of exponential type at most 1 in the space L^p on $\mathbb{R}$ with the power weight $|x|^{2\alpha+1}$. Moreover, we give explicit bounds of the form

$$n^{(2\alpha+2)/p}\mathcal{L}_p \leq \mathcal{C}_\alpha(n) \leq (n + 2s_{p,\alpha})^{(2\alpha+2)/p}\mathcal{L}_p, \quad n \geq 0,$$

from which the given asymptotics follows. These bounds allow one to refine the known estimates of the Nikol'skii constants. We consider such approach using the example of the algebraic Nikol'skii constant for $\alpha = 0$. Here we apply the characterization of the extremal polynomials from the works of D. Amir and Z. Ziegler, V. Arestov and M. Deikalova. Our statements generalize the known results of S. B. Stechkin ($p = 1$), E. Levin and D. Lubinsky ($p > 0$) in the trigonometric case for $\alpha = -1/2$, and M. I. Ganzburg in the algebraic case for $\alpha = 0$. For half-integer $\alpha = d/2 - 1$ and $p \geq 1$, our asymptotics can be derived from the asymptotic of the multidimensional Nikol'skii constant for spherical polynomials in the space L^p on the sphere $\mathbb{S}^d$ proved by F. Dai, D. Gorbachev, and S. Tikhonov. Our proof is much simpler, but it does not cover the case $p < 1$.

Keywords: Nikol'skii inequality, sharp constant, asymptotic behavior, trigonometric polynomial, algebraic polynomial, entire function of exponential type, Gegenbauer weight.

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