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CONVEXITY AND MONOTONE LINEAR CONNECTIVITY OF SETS WITH A CONTINUOUS METRIC PROJECTION IN THREE-DIMENSIONAL SPACES

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A continuous curve $k(\cdot)$ in a normed linear space X is called monotone if the function $f(k(\tau))$ is monotone with respect to τ for any extreme functional f of the unit dual sphere S^*. A closed set is monotone path-connected if any two points from it can be connected by a continuous monotone curve lying in this set. We prove that in a three-dimensional Banach space any closed set with lower semi-continuous metric projection is monotone path-connected if and only if the norm of the space is either cylindrical or smooth. This result partially extends a recent result of the author of this paper and B. B. Bednov, who characterized the three-dimensional spaces in which any Chebyshev set is monotone path-connected. We show that in a finite-dimensional Banach space any closed set with lower semi-continuous (continuous) metric projection is convex if and only if the space is smooth. A number of new properties of strict suns in three-dimensional spaces with cylindrical norm is put forward. It is shown that in a three-dimensional space with cylindrical norm a closed set M with lower semi-continuous metric projection is a strict sun. Moreover, such a set M has contractible intersections with closed balls and possesses a continuous selection of the metric projection operator. Our analysis depends substantially on the novel machinery of approximation of the unit sphere by polytopes built from tangent directions to the unit sphere.

Keywords: set with continuous metric projection, Chebyshev set, sun, monotone path-connected set.

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