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ON FINITE SIMPLE GROUPS OF EXCEPTIONAL LIE TYPE OVER FIELDS OF DIFFERENT CHARACTERISTICS WITH COINCIDING PRIME GRAPHS

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Suppose that G is a finite group, $\pi(G)$ is the set of prime divisors of its order, and $\omega(G)$ is the set of orders of its elements. A graph with the following adjacency relation is defined on $\pi(G)$: different vertices r and s from $\pi(G)$ are adjacent if and only if $rs \in \omega(G)$. This graph is called the *Gruenberg–Kegel graph* or the *prime graph* of G and is denoted by $GK(G)$. In A. V. Vasil'ev's Question 16.26 from the "Kourovka Notebook," it is required to describe all pairs of nonisomorphic finite simple nonabelian groups with identical Gruenberg–Kegel graphs. M. Hagie (2003) and M. A. Zvezdina (2013) gave such a description in the case where one of the groups coincides with a sporadic group and an alternating group, respectively. The author (2014) solved this question for pairs finite simple groups of Lie type over fields of the same characteristic. In the present paper we prove the following theorem.

Theorem. Let G be a finite simple group of exceptional Lie type over a field with q elements, and let G_1 be a finite simple group of Lie type over a field with q elements nonisomorphic to G, where q and q_1 are coprime. If $GK(G) = GK(G_1)$, then one of the following statements holds:

(1) $\{G, G_1\} = \{G_2(3), A_1(13)\}$;
(2) $\{G, G_1\} = \{{}^2F_4(2)', A_3(3)\}$;
(3) $\{G, G_1\} = \{{}^3D_4(q), A_2(q_1)\}$, where $(q_1 - 1)_3 \neq 3$ and $q_1 + 1 \neq 2^{k_1}$;
(4) $\{G, G_1\} = \{{}^3D_4(q), A_4^{\pm}(q_1)\}$, where $(q_1 \mp 1)_5 \neq 5$;
(5) $\{G, G_1\} = \{G_2(q), G_2(q_1)\}$, where q and q_1 are not powers of the number 3;
(6) $\{G, G_1\}$ is one of the pairs $\{F_4(q), F_4(q_1)\}$, $\{{}^3D_4(q), {}^3D_4(q_1)\}$, and $\{E_8(q), E_8(q_1)\}$.

The existence of pairs of groups in statements (3)–(6) is unknown.

Keywords: finite simple exceptional group of Lie type, spectrum, prime graph.

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