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# ASYMPTOTIC BEHAVIOR OF SMALL-TIME REACHABLE SETS OF NONLINEAR SYSTEMS WITH ISOPERIMETRIC CONSTRAINTS

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We study the problem of an approximate description of reachable sets over small time intervals for affine-control systems with isoperimetric control constraints. An isoperimetric constraint is understood as an integral constraint of inequality type with the integrand depending on the control parameters and state variables of the system. Previously, a similar problem was considered under the assumption that the integrand depends only on the control parameters and is a positive definite quadratic form in these parameters. In this case, it was shown that, under certain conditions imposed on the controllability Gramian of the linearized system, the reachable set is convex and asymptotically close in shape to an ellipsoid in the state space for a sufficiently small length of the time interval. This ellipsoid is the reachable set of the system linearized along the trajectory corresponding to the null control. In this paper, it is proved that, under a slight strengthening of the conditions imposed on the controllability Gramian, this result remains valid if the integrand defining the isoperimetric constraints has the form of the sum of a positive definite quadratic form in the control parameters and a nonnegative function of the state variables. This asymptotic representation holds, in particular, for a fairly wide class of second-order systems affine in the control under the condition that the linearized system is completely controllable. The proof is based on the results of the theory of strongly convex sets and functions.

Keywords: control system, isoperimetric constraints, reachable set, asymptotics, controllability Gramian.

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