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ASYMPTOTICS OF A SOLUTION TO A PROBLEM OF OPTIMAL BOUNDARY CONTROL WITH TWO SMALL COSUBORDINATE PARAMETERS

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We consider a problem of optimal boundary control for solutions of an elliptic type equation in a bounded domain with smooth boundary with a small coefficient at the Laplace operator, a small coefficient, cosubordinate with the first, at the boundary condition, and integral constraints on the control:

$$\begin{cases} \mathcal{L}_\varepsilon := -\varepsilon^2 \Delta z + a(x)z = f(x), \quad x \in \Omega, \quad z \in H^1(\Omega), \\ l_{\varepsilon,\beta} z := \varepsilon^\beta \dfrac{\partial z}{\partial n} = g(x) + u(x), \quad x \in \Gamma, \end{cases}$$

$$J(u) := \|z - z_d\|^2 + \nu^{-1} |||u|||^2 \to \inf, \quad u \in \mathcal{U},$$

where $0 < \varepsilon \ll 1$, $\beta \geqslant 0$, $\beta \in \mathbb{Q}$, $\nu > 0$, $H^1(\Omega)$ is the Sobolev function space, $\partial z/\partial n$ is the derivative of z at the point $x \in \Gamma$ in the direction of the outer (with respect to the domain Ω) normal,

$$a(\cdot), f(\cdot) \in C^\infty(\overline{\Omega}), \quad g(\cdot) \in C^\infty(\Gamma), \quad \forall\, x \in \overline{\Omega} \quad a(x) \geqslant \alpha^2 > 0,$$
$$\mathcal{U} = \mathcal{U}_1, \quad \mathcal{U}_r := \{u(\cdot) \in L_2(\Gamma) \colon |||u||| \leqslant r\}.$$

Here $\|\cdot\|$ and $|||\cdot|||$ are the norms in the spaces $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. We find the complete asymptotic expansion of the solution of the problem in the powers of the small parameter in the case where $0 < \beta < 3/2$.

Keywords: singular problems, optimal control, boundary value problems for systems of partial differential equations, asymptotic expansions.

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