ON THE RECOGNIZABILITY OF SPORADIC SIMPLE GROUPS
RU, HN, FI_{22}, HE, $M^C L$, **AND** CO_3
BY THE GRUENBERG–KEGEL GRAPH

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The Gruenberg–Kegel graph (prime graph) $\Gamma(G)$ of a finite group G is a graph in which the vertices are the prime divisors of the order of G and two distinct vertices p and q are adjacent if and only if G contains an element of order pq. The problem of recognition of finite groups by the Gruenberg–Kegel graph is of great interest in the finite group theory. For a finite group G, $h_\Gamma(G)$ denotes the number of all pairwise nonisomorphic finite groups H such that $\Gamma(H) = \Gamma(G)$ (if the set of such groups H is infinite, we write $h_\Gamma(G) = \infty$). A group G is called n-recognizable by the Gruenberg–Kegel graph if $h_\Gamma(G) = n < \infty$, recognizable the Gruenberg–Kegel graph if $h_\Gamma(G) = 1$, and unrecognizable the Gruenberg–Kegel graph if $h_\Gamma(G) = \infty$. We say that the problem of recognizability by the Gruenberg–Kegel graph is solved for a finite group G if the value $h_\Gamma(G)$ is found. For a finite group G unrecognizable by the Gruenberg–Kegel graph, the question of the (normal) structure of finite groups with the same Gruenberg–Kegel graph as G is also of interest. In 2003, M. Hagie investigated the structure of finite groups having the same Gruenberg–Kegel graph as some sporadic simple group. In particular, she gave first examples of finite groups recognizable by the Gruenberg–Kegel graph; they were the sporadic simple groups J_1, M_{22}, M_{23}, M_{24}, and Co_2. However, that investigation was not completed. In 2006, A. V. Zavarnitsine established that the group J_4 is recognizable by the Gruenberg–Kegel graph. The unrecognizability of the sporadic groups M_{12} and J_2 was known previously; it follows from the unrecognizability of these groups by the spectrum. In the present paper, we continue Hagie's study and use her results. For any sporadic simple group S isomorphic to Ru, HN, Fi_{22}, He, $M^c L$, or Co_3, we find all finite groups having the same Gruenberg–Kegel graph as S. Thus, for these six groups, we complete Hagie's investigation and, in particular, solve the problem of recognizability by the Gruenberg–Kegel graph.

Keywords: finite group, simple group, sporadic group, spectrum, Gruenberg–Kegel graph, recognition by the Gruenberg–Kegel graph.

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