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SOME SCHURIAN ASSOCIATION SCHEMES RELATED TO SUZUKI AND REE GROUPS

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An *association scheme* is a pair $(\Omega,\mathcal{R})$ consisting of a finite set Ω and a set $\mathcal{R}=\{R_0,R_1\ldots,R_s\}$ of binary relations on Ω satisfying the following conditions: (1) $\mathcal{R}$ is a partition of the set Ω^2; (2) $\{(x,x)\mid x\in\Omega\}\in\mathcal{R}$; (3) ${R_t}^T=\{(y,x)\mid (x,y)\in R_t\}\in\mathcal{R}$ for all $0\le t\le s$; (4) for all $0\le i,j,t\le s$, there exist constants c_{ij}^t (called the *intersection numbers* of the scheme) such that $c_{ij}^t=|\{z\in\Omega|(x,z)\in R_i,(z,y)\in R_j\}|$ for any pair $(x,y)\in R_t$. An association scheme $(\Omega,\mathcal{R})$ is called *Schurian* if, for some permutation group on Ω, the set of orbitals of this group on Ω coincides with $\mathcal{R}$. This work is devoted to the study of Schurian association schemes related to Suzuki groups $Sz(q)$ and Ree groups ${}^2G_2(q)$ with $q>3$ for which some graphs of their basic relations are antipodal distance-regular graphs of diameter 3. Assume that G is one of the mentioned groups, $r=(q-1)_{2'}$, B is a Borel subgroup of G, U is a unipotent subgroup of G contained in B, K is a subgroup of B with index r, g is an involution in $G-B$, and f is an element of order r in $B\cap B^g$. Let Ω be the set of the right K-cosets of G, and put $h_i=f^i$ and $h_{r+i}=gf^i$ for all $i\in\{0,\ldots,r-1\}$. Denote by $\mathcal{R}$ the set $\{R_0,R_1,\ldots,R_{2r-1}\}$ of binary relations on Ω defined for each $t\in\{0,1,\ldots,2r-1\}$ by the rule: $(Kx,Ky)\in R_t$ if and only if xy^{-1} is contained in the double coset Kh_tK. We prove that $\mathcal{X}=(\Omega,\mathcal{R})$ is a Schurian association scheme and its set of basic relations coincides with the set of orbitals of G on Ω. We find that the intersection number c_{ij}^t, where $0\le i,j,t\le 2r-1$, of the scheme $\mathcal{X}$ is $|U|$ if $t\le r-1$, $i,j\ge r$, and $j-i\equiv t \pmod r$; $(|U|-1)/r$ if $i,j,t\ge r$; 1 if either $t\le r-1$, $i,j\le r-1$, and $i+j\equiv t \pmod r$, or $i\le r-1$, $t,j\ge r$, and $j-i\equiv t \pmod r$, or $t,i\ge r$, $j\le r-1$, and $i+j\equiv t$ $(\mathrm{mod}\ r)$; and 0 in the remaining cases, where $|U|=q^2$ if $G=Sz(q)$ and $|U|=q^3$ if $G={}^2G_2(q)$. As a corollary, we find the structural parameters $m_{h_t}(h_i,h_j)=|\{Kx\in\Omega|\ Kx\subseteq Kh_i^{-1}Kh_t\cap Kh_jK\}|$ of the Hecke algebra $\mathbb{C}(K\backslash G/K)$ of G with respect to K. Namely, we show that $m_{h_t}(h_i,h_j)$ is exactly the intersection number c_{ij}^t of the scheme $\mathcal{X}$ for all $0\le i,j,t\le 2r-1$. By definition, the graph of the basic relation R_t with $t\ge r$ of $\mathcal{X}$ is equivalent to the coset graph $\Gamma(G,K,Kh_tK)$ of G with respect to K and the element h_t and, as is known, is an antipodal distance-regular graph of diameter 3 with intersection array $\{|U|,(|U|-1)(r-1)/r,1;1,(|U|-1)/r,|U|\}$. The latter fact was proved in the author's earlier paper, where we proposed a technique for studying the graphs $\Gamma(G,K,Kh_tK)$; the technique is based on analyzing the mutual distribution of the neighborhoods of vertices. In the present work, we prove the distance regularity of these graphs as a corollary of the properties of the scheme $\mathcal{X}$.

Keywords: Schurian association scheme, distance-regular graph, antipodal graph.

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