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BERNSTEIN–SZEGŐ INEQUALITY FOR TRIGONOMETRIC POLYNOMIALS IN THE SPACE L_0

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Inequalities of the form $\|f_n^{(\alpha)}\cos\theta+\tilde f_n^{(\alpha)}\sin\theta\|_p\le B_n(\alpha,\theta)_p\|f_n\|_p$ for classical derivatives of order $\alpha\in\mathbb{N}$ and Weyl derivatives of real order $\alpha\ge 0$ of trigonometric polynomials f_n of order $n\ge 1$ and their conjugates for real θ and $0\le p\le\infty$ are called Bernstein–Szegő inequalities. They are generalizations of the classical Bernstein inequality ($\alpha=1$, $\theta=0$, $p=\infty$). Such inequalities have been studied for more than 90 years. The problem of studying the Bernstein–Szegő inequality consists in analyzing the properties of the best (the smallest) constant $B_n(\alpha,\theta)_p$, its exact value, and extremal polynomials for which this inequality turns into an equality. G. Szegő (1928), A. Zygmund (1933), and A. I. Kozko (1998) showed that, in the case $p\ge 1$ for real $\alpha\ge 1$ and any real θ, the best constant $B_n(\alpha,\theta)_p$ is n^α. For $p=0$, Bernstein–Szegő inequalities are of interest at least because the constant $B_n(\alpha,\theta)_p$ is the largest for $p=0$ over $0\le p\le\infty$. In 1981, V. V. Arestov proved that, for $r\in\mathbb{N}$ and $\theta=0$, the Bernstein inequality is true with the constant n^r in the spaces L_p, $0\le p<1$; i.e., $B_n(r,0)_p=n^r$. In 1994, he proved that, for $p=0$ and the derivative of the conjugate polynomial of order $r\in\mathbb{N}\cup\{0\}$, i.e., for $\theta=\pi/2$, the exact constant grows exponentially in n; more precisely, $B_n(r,\pi/2)_0=4^{n+o(n)}$. In two recent papers of the author (2018), a similar result was obtained for Weyl derivatives of positive noninteger order for any real θ. In the present paper, we prove that the formula $B_n(\alpha,\theta)_0=4^{n+o(n)}$ holds for derivatives of nonnegative integer orders α and any real $\theta\ne\pi k$, $k\in\mathbb{Z}$.

Keywords: trigonometric polynomial, conjugate polynomial, Weyl derivative, Bernstein–Szegő inequality, space L_0.

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