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ABSTRACT CONVEXITY OF FUNCTIONS WITH RESPECT TO THE SET OF LIPSCHITZ (CONCAVE) FUNCTIONS

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The paper is devoted to the abstract $\mathcal{H}$-convexity of functions (where $\mathcal{H}$ is a given set of elementary functions) and its realization in the cases when $\mathcal{H}$ is the space of Lipschitz functions or the set of Lipschitz concave functions. We introduce the notion of regular $\mathcal{H}$-convex functions. These are functions representable as the upper envelopes of the set of their maximal (with respect to the pointwise ordering) $\mathcal{H}$-minorants. As a generalization of the global subdifferential of a convex function, we introduce the set of maximal support $\mathcal{H}$-minorants at a point and the set of lower $\mathcal{H}$-support points. Using these tools, we formulate necessary as well as sufficient conditions for global minima of nonsmooth functions. In the second part of the paper, the abstract notions of $\mathcal{H}$-convexity are realized in the specific cases when functions are defined on a metric or normed space X and the set of elementary functions is the space $\mathcal{L}(X,\mathbb{R})$ of Lipschitz functions or the set $\mathcal{L}\widehat{C}(X,\mathbb{R})$ of Lipschitz concave functions, respectively. An important result of this part of the paper is the proof of the fact that, for a lower semicontinuous function bounded from below by a Lipschitz function, the set of its lower $\mathcal{L}$-support points and the set of lower $\mathcal{L}\widehat{C}$-support points coincide and are dense in the effective domain of the function. These results extend the known Brøndsted–Rockafellar theorem on the existence of a subdifferential of convex lower semicontinuous functions to the wider class of lower semicontinuous functions and go back to the Bishop–Felps theorem on the density of support points in the boundary of a closed convex set, which is one of most important results of classical convex analysis.

Keywords: abstract convexity, support minorants, support points, global minimum, semicontinuous functions, Lipschitz functions, concave Lipschitz functions, density of support points.

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