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NIKOL'SKII–BERNSTEIN CONSTANTS FOR ENTIRE FUNCTIONS OF EXPONENTIAL SPHERICAL TYPE IN WEIGHTED SPACES

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We study the exact constant in the Nikol'skii–Bernstein inequality $\|Df\|_q \le C\|f\|_p$ on the subspace of entire functions f of exponential spherical type in the space $L^p(\mathbb{R}^d)$ with a power-type weight v_κ. For the differential operator D, we take a nonnegative integer power of the Dunkl Laplacian Δ_κ associated with the weight v_κ. This situation encompasses the one-dimensional case of the space $L^p(\mathbb{R}_+)$ with the power weight $t^{2\alpha+1}$ and Bessel differential operator. Our main result consists in the proof of an equality between the multidimensional and one-dimensional weight constants for $1 \le p \le q = \infty$. For this, we show that the norm $\|Df\|_\infty$ can be replaced by the value $Df(0)$, which was known only in the one-dimensional case. The required mapping of the subspace of functions, which actually reduces the problem to the radial and, hence, one-dimensional case, is implemented by means of the positive operator of generalized Dunkl translation T_κ^t. We prove its new property of analytic continuation in the variable t. As a consequence, we calculate the weighted Bernstein constant for $p = q = \infty$, which was known in exceptional cases only. We also find some estimates of the constant and give a short list of open problems.

Keywords: Nikol'skii–Bernstein inequality, exact constant, entire function of exponential spherical type, power-type weight, Dunkl Laplacian.

REFERENCES

1. Arestov V.V. On inequalities of S.N. Bernstein for algebraic and trigonometric polynomials. *Soviet Math. Dokl.*, 1979, vol. 20, no. 3, pp. 600–603.
2. Arestov V.V. Inequality of different metrics for trigonometric polynomials. *Math. Notes*, 1980, vol. 27, no. 4, pp. 265–269. doi: 10.1007/BF01140526 .
3. Arestov V., Babenko A., Deikalova M., Horváth Á. Nikol'skii inequality between the uniform norm and integral norm with Bessel weight for entire functions of exponential type on the half-line. *Anal. Math.*, 2018, vol. 44, no. 1, pp. 21–42. doi: 10.1007/s10476-018-0103-6 .
4. Dai F., Gorbachev D., Tikhonov S. Nikolskii constants for polynomials on the unit sphere. Available at: *arXiv*:1708.09837 [math.CA], 2017, 21 p.
5. Ganzburg M. Sharp constants of approximation theory. I. Multivariate Bernstein–Nikolskii type inequalities. *arXiv*:1901.04400 [math.CA], 2019, 19 p.
6. Ganzburg M.I., Tikhonov S.Yu. On sharp constants in Bernstein–Nikolskii inequalities. *Constr. Approx.*, 2017, vol. 45, no. 3, pp. 449–466. doi: 10.1007/s00365-016-9363-1 .
7. Gorbachev D.V., Dobrovolskii N.N. Nikolskii constants in $L^p(\mathbb{R}, |x|^{2\alpha+1}\,dx)$ spaces. *Chebyshevskii Sbornik*, 2018, vol. 19, no. 2, pp. 67–79 (in Russian). doi: 10.22405/2226-8383-2018-19-2-67-79 .
8. Gorbachev D.V., Martyanov I.A. On interrelation of Nikolskii Constants for trigonometric polynomials and entire functions of exponential type. *Chebyshevskii Sbornik*, 2018, vol. 19, no. 2, pp. 80–89 (in Russian). doi: 10.22405/2226-8383-2018-19-2-80-89 .
9. Gorbachev D.V., Ivanov V.I., Tikhonov S.Y. Positive L^p-bounded Dunkl-type generalized translation operator and its applications. *Constr. Approx.*, 2018, pp. 1–51. doi: 10.1007/s00365-018-9435-5 .
10. Gorbachev D.V., Ivanov V.I. Fractional smoothness in L^p with Dunkl weight and its applications. *arXiv*:1812.04946 [math.CA], 2019, 28 p.
11. Ivanov V.A. Precise results in the problem of the Bernstein – Nikol'skij inequality on compact symmetric Riemannian spaces of rank 1. *Proc. Steklov Inst. Math.*, Investigations in the theory of differentiable functions of many variables and its applications, part 14, 1993, vol. 194, pp. 115–124.

12. de Jeu M.F.E. Paley–Wiener theorems for the Dunkl transform. *Trans. Amer. Math. Soc.*, 2006, vol. 358, pp. 4225–4250. doi: 10.1090/S0002-9947-06-03960-2 .
13. Kamzolov A.I. On Riesz's interpolational formula and Bernshtein's inequality for functions on homogeneous spaces. *Math. Notes*, 1974, vol. 15, no. 6, pp. 576–582. doi: 10.1007/BF01152838 .
14. Mejjaoli H., Trimèche K. On a mean value property associated with the Dunkl Laplacian operator and applications. *Integral Transform. Spec. Funct.*, 2001, vol. 12, pp. 279–302. doi: 10.1080/10652460108819351 .
15. Levin E., Lubinsky D. Asymptotic behavior of Nikolskii constants for polynomials on the unit circle. *Comput. Methods Funct. Theory*, 2015, vol. 15, pp. 459–468. doi: 10.1007/s40315-015-0113-3 .
16. Nessel R., Wilmes G. Nikolskii-type inequalities for trigonometric polynomials and entire functions of exponential type. *J. Austral. Math. Soc.*, 1978, vol. 25, no. 1, pp. 7–18. doi: 10.1017/S1446788700038878 .
17. Nikol'skii S.M. *Approximation of functions of several variables and embedding theorems.* Berlin; N Y: Springer-Verlag, 1975, 420 p. doi: 10.1007/978-3-642-65711-5 . Original Russian text (2nd ed.) published in Nikol'skii S.M. *Priblizhenie funktsii mnogikh peremennykh i teoremy vlozheniya.* Moscow: Nauka Publ., 1977, 480 p.
18. Platonov S.S. Bessel harmonic analysis and approximation of functions on the half-line. *Izvestiya: Math.*, 2007, vol. 71, no. 5, pp. 1001–1048. doi: 10.1070/IM2007v071n05ABEH002379 .
19. Rösler M. A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.*, 2003, vol. 355, pp. 2413–2438. doi: 10.1090/S0002-9947-03-03235-5 .

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