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KOLMOGOROV WIDTHS OF SOBOLEV CLASSES ON A CLOSED INTERVAL WITH CONSTRAINTS ON THE VARIATION

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We study the problem of estimating Kolmogorov widths in $L_q[0,1]$ for the Lipschitz classes of functions with fixed values at several points: $\tilde{M} = \{f \in AC[0,1],\ \|\dot{f}\|_\infty \le 1,\ f(j/s) = y_j,\ 0 \le j \le s\}$. Applying well-known results about the widths of Sobolev classes, it is easy to obtain order estimates up to constants depending on q and $y_1, \ldots, y_n$. Here we obtain order estimates up to constants depending only on q. To this end, we estimate the widths of the intersection of two finite-dimensional sets: a cube and a weighted Cartesian product of octahedra. If we take the unit ball of l_p^n instead of the cube, we get a discretization of the problem on estimating the widths of the intersection of the Sobolev class and the class of functions with constraints on their variation: $M = \{f \in AC[0,1] :\ \|\dot{f}\|_{L_p[0,1]} \le 1,\ \|\dot{f}\|_{L_1[(j-1)/s,\, j/s]} \le \varepsilon_j/s,\ 1 \le j \le s\}$. For sufficiently large n, order estimates are obtained for the widths of these classes up to constants depending only on p and q. If $p > q$ or $p > 2$, then these estimates have the form $\varphi(\varepsilon_1, \ldots, \varepsilon_s)n^{-1}$, where $\varphi(\varepsilon_1, \ldots, \varepsilon_s) \to 0$ as $(\varepsilon_1, \ldots, \varepsilon_s) \to 0$ (explicit formulas for φ are given in the paper). If $p \le q$ and $p \le 2$, then the estimates have the form n^{-1} (hence, the constraints on the variation do not improve the estimate for the widths). The upper estimates are proved with the use of Galeev's result on the intersection of finite-dimensional balls, whereas the proof of the lower estimates is based on a generalization of Gluskin's result on the width of the intersection of a cube and an octahedron.

Keywords: Kolmogorov widths, Sobolev classes, interpolation classes.

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