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ON THE APPROXIMATION OF THE HILBERT TRANSFORM

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The article is devoted to the approximation of the Hilbert transform $(Hu)(t) = \dfrac{1}{\pi}\displaystyle\int_R \dfrac{u(\tau)}{t-\tau}d\tau$ of functions $u \in L_2(R)$ by operators of the form $(H_\delta u)(t) = \dfrac{1}{\pi}\displaystyle\sum_{k=-\infty}^{\infty} \dfrac{u(t+(k+1/2)\delta)}{-k-1/2}$, $\delta > 0$. The main results are the following statements.

Theorem 1. For any $\delta > 0$ the operators H_δ are bounded in the space $L_p(R)$, $1 < p < \infty$, and

$$\|H_\delta\|_{L_p(R)\to L_p(R)} \le \left\|\tilde{h}\right\|_{l_p\to l_p},$$

where $\tilde{h}$ is the modified discrete Hilbert transform defined by the equality

$$\tilde{h}(b) = \left\{\left(\tilde{h}(b)\right)_n\right\}_{n\in Z}, \quad \left(\tilde{h}(b)\right)_n = \sum_{m\in Z} \frac{b_m}{n-m-1/2}, \quad n \in Z, \quad b = \{b_n\}_{n\in Z} \in l_1.$$

Theorem 2. For any $\delta > 0$ and $u \in L_p(R)$, $1 < p < \infty$, the following inequality holds:

$$H_\delta(H_\delta u)(t) = -u(t).$$

Theorem 3. For any $\delta > 0$ the sequence of operators $\left\{H_{\delta/n}\right\}_{n\in N}$ strongly converges to the operator H in $L_2(R)$; i.e., the following inequality holds for any $u \in L_2(R)$:

$$\lim_{n\to\infty} \left\|H_{\delta/n}u - Hu\right\|_{L_2(R)} = 0.$$

Keywords: Hilbert transform, singular integral, approximation, discrete Hilbert transform.

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