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MULTIVARIATE VERSION OF TURAN'S TYPE INEQUALITY AND ITS APPLICATIONS TO THE ESTIMATION OF UNIFORM MODULI OF SMOOTHNESS OF PERIODIC FUNCTIONS

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The following results are proved in the paper.

Theorem 1. *Let* $m \geq 1,\ f \in L_1(\mathbb{T}^m),\ l,k \in \mathbb{N},\ l > m,\ \rho = l-(k+m)$, *and* $\sum_{n=1}^{\infty} n^{m-1}\omega_l(f;d/n)_{1,m} < \infty$. *Then* f *is equivalent to some function* $\psi \in C(\mathbb{T}^m)$ *and*

$$\text{(a)}\quad \omega_k\Big(\psi;\frac{d}{n}\Big)_{\infty,m} \leq C_1(k,l,m)\bigg\{\sum_{\nu=n+1}^{\infty}\nu^{m-1}\omega_l\Big(f;\frac{d}{\nu}\Big)_{1,m} + \chi(\rho)n^{-k}\sum_{\nu=1}^{n}\nu^{k+m-1}\omega_l\Big(f;\frac{d}{\nu}\Big)_{1,m}\bigg\},\quad n\in\mathbb{N},$$

where $\omega_l(f;\delta)_{1,m}$ *is the* l *th-order complete modulus of smoothness of* f, $\omega_k(\psi;\delta)_{\infty,m}$ *is the* k *th-order complete modulus of smoothness of* ψ, $\mathbb{T}^m = (-\pi,\pi]^m$, $d = \pi m^{1/2}$, $\chi(t) = 0$ *for* $t \leq 0$, *and* $\chi(t) = 1$ *for* $t > 0$.

In the case $l = k+m$ ($\Rightarrow \chi(\rho) = 0$), the proof of estimate (a) relies substantially on the inequality

$$\text{(b)}\quad n^{-k}\max_{|\alpha|=k}\Big\|\frac{\partial^{|\alpha|}T_{n,\dots,n;1}(f;x)}{\partial x^\alpha}\Big\|_{\infty,m} \leq C_2(k,m)n^m\omega_{k+m}\Big(f;\frac{d}{n+1}\Big)_{1,m},\quad n\in\mathbb{N},$$

where $T_{n,\dots,n;1}(f;x_1,\dots,x_m)$ is a polynomial of best $L_1(\mathbb{T}^m)$-approximation to f of order $n\in\mathbb{N}$ with respect to the variable x_i $(i=\overline{1,m})$ and $\alpha=(\alpha_1,\dots,\alpha_m)$, $\alpha_j\in\mathbb{Z}_+$ $(j=\overline{1,m})$, is a multiindex of length $|\alpha|=k$. Inequality (b) is proved by using a multivariate version of Turan's type inequality: for each trigonometric polynomial $t_{n_1,\dots,n_m}(x_1,\dots,x_m)$ of order $n_i\in\mathbb{N}$ with respect to the variable x_i $(i=\overline{1,m})$, we have the inequality

$$\text{(c)}\quad \Big\|\frac{\partial^k t_{n_1,\dots,n_m}(x)}{\partial x^\alpha}\Big\|_{\infty,m} \leq \Big(\frac{\pi}{2}\Big)^m\Big\|\frac{\partial^{k+m}t_{n_1,\dots,n_m}(x_1,\dots,x_m)}{\partial x_1^{\alpha_1+1}\dots\partial x_m^{\alpha_m+1}}\Big\|_{1,m},$$

which follows directly from a similar inequality (with $k=0$ in inequality (c)) but holds under the conditions $\frac{1}{2\pi}\int_0^{2\pi} t_{n_1,\dots,n_i,\dots,n_m}(x_1,\dots,x_i-y_i,\dots,x_m)\,dy_i = 0$, $i=\overline{1,m}$.

Estimate (a) is order-sharp in the class $H^l_{1,m}[\omega] = \{f\in L_1(\mathbb{T}^m):\ \omega_l(f;\delta)_{1,m}\leq\omega(\delta),\ \delta\in(0,d]\}$, where $\omega\in\Omega_l(0,d]$ is the class of functions $\omega=\omega(\delta)$ defined on $(0,d]$ and satisfying the conditions $0<\omega(\delta)\downarrow 0$ $(\delta\downarrow 0)$ and $\delta^{-l}\omega(\delta)\downarrow$ $(\delta\uparrow)$.

Theorem 2. *Let* $m\geq 1,\ l,k\in\mathbb{N},\ l>m,\ \rho=l-(k+m),\ \omega\in\Omega_l(0,d]$, *and* $\sum_{n=1}^{\infty}n^{m-1}\omega(d/n)<\infty$. *Then*

$$\sup\Big\{\omega_k\Big(\psi;\frac{d}{n}\Big)_{\infty,m}:\ f\in H^l_{1,m}[\omega]\Big\} \asymp \sum_{\nu=n+1}^{\infty}\nu^{m-1}\omega\Big(\frac{d}{\nu}\Big) + \chi(\rho)n^{-k}\sum_{\nu=1}^{n}\nu^{k+m-1}\omega\Big(\frac{d}{\nu}\Big),\quad n\in\mathbb{N},$$

where ψ *is the corresponding function from the class* $C(\mathbb{T}^m)$ *equivalent to* $f\in H^l_{1,m}[\omega]$.

Keywords: complete modulus of smoothness, multivariate version of Turan's type inequality, inequalities between moduli of smoothness of various order in different metrics, order-sharp inequality on a class.

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