TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN

Vol. 25 No. 1 **2019**

MSC: 93C10, 93C23

DOI: 10.21538/0134-4889-2019-25-1-229-244

ON A CLASS OF PROBLEMS OF OPTIMAL IMPULSE CONTROL FOR A CONTINUITY EQUATION

M. V. Staritsyn, N. I. Pogodaev

We consider an impulse control problem for a special class of distributed dynamical systems. Such systems result from a relaxation (extension of the set of control processes) of a continuity equation driven by a vector field affine in the control, when there are only integral constraints on the input signals. Problems of this kind appear in the theory of ensemble control and control of multi-agent systems and systems with uncertain initial data. Prior to relaxation, the states of the system may be arbitrarily close to discontinuous curves in the space of probability measures, which leads to the unsolvability of the corresponding extremal problem. The relaxation produces a well-posed optimal control problem for generalized solutions of the continuity equation, which are measure-valued curves with bounded variation. Generalized solutions are described by means of a discontinuous time change in the trajectories of the characteristic system. Some function-theoretic properties of these solutions are studied, and their representation in terms of measure differential equations is obtained. The main result is a necessary optimality condition in the form of the maximum principle for the relaxed problem. Finally, we discuss the possibilities of applying the results for the development of numerical algorithms.

Keywords: multi-agent systems, continuity equation, impulse-trajectory relaxation, ensemble control, impulse control, optimal control, maximum principle, numerical algorithms for optimal control.

REFERENCES

1. Ovseevich A. I., Fedorov A. K. Asymptotically optimal feedback control for a system of linear oscillators. *Dokl. Akad. Nauk*, 2013, vol. 88, no. 2, pp. 613–617. doi: 10.1134/S106456241305013X .
2. Li J.-S. Ensemble control of finite-dimensional time-varying linear systems. *IEEE Trans. Autom. Control*, 2011, vol. 56, no. 2, pp. 345–357.
3. Conservation laws in the modeling of moving crowds. In Rinaldo M. Colombo, Mauro Garavello, Magali Lécureux-Mercier, Nikolay Pogodaev (eds.). *Hyperbolic problems: theory, numerics, applications*, Am. Inst. Math. Sci. (AIMS), Springfield, MO, 2014. vol. 8, AIMS Ser. Appl. Math., pp. 467–474.
4. Fornasier M., Solombrino F. Mean field optimal control. *ESAIM: Control, Optimization and Calculus of Variations*, 2014, vol. 20, no. 4, pp. 1123–1152. doi: 10.1051/cocv/2014009 .
5. Marigonda A., Quincampoix M. Mayer control problem with probabilistic uncertainty on initial positions. *J. Dierential Eq.*, 2018, vol. 264, no. 5, pp. 3212–3252. doi: 10.1016/j.jde.2017.11.014 .
6. Li J.-S., Khaneja N. Ensemble control of Bloch equations., *IEEE Trans. Autom. Control*, 2009, vol. 54, no. 3, pp. 528–536. doi: 10.1109/TAC.2009.2012983 .
7. Arutyunov A., Karamzin D., Pereira F. L. On a generalization of the impulsive control concept: controlling system jumps. *Discrete Contin. Dyn. Syst.*, 2011, vol. 29, no. 2, pp. 403–415. doi: 10.3934/dcds.2011.29.403 .
8. Bressan Jr. A., Rampazzo F. Impulsive control systems without commutativity assumptions. *J. Optim. Theory Appl.*, 1994, vol. 81, no. 3, pp. 435–457. doi: 10.1007/BF02193094 .
9. Dykhta V.A., Samsonyuk O.N. *Optimal'noe impul'snoe upravlenie s prilozheniyami* [Optimal impulse equation with applications]. Moscow, Fizmatlit Publ., 2000, 256 p. ISBN: 5-9221-0097-1 .
10. Gurman V. Extensions and global estimates for evolutionary discrete control systems. In: Kurzhanski A., Lasiecka I. (eds.) *Modelling and inverse problems of control for distributed parameter systems.* (Laxenburg, 1989). Ser. Lecture Notes in Control and Inform. Sci., vol. 154. Berlin: Springer, 1991, pp. 16–21. doi: 10.1007/BFb0044479 .

11. Karamzin D.Y., V.A. de Oliveira, Pereira F.L., Silva G.N. On the properness of an impulsive control extension of dynamic optimization problems. *ESAIM Control Optim. Calc. Var.*, 2015, vol. 21, no. 3, pp. 857–875. doi: 10.1051/cocv/2014053 .
12. Miller B. M., Rubinovich E. Y. *Impulsive control in continuous and discrete-continuous systems.* N Y: Kluwer Academic/Plenum Publ., 2003, 447 p. doi: 10.1007/978-1-4615-0095-7 .
13. Motta M., Rampazzo F. Space-time trajectories of nonlinear systems driven by ordinary and impulsive controls. *Differential Integral Eq.*, 1995, vol. 8, no. 2, pp. 269–288.
14. Rishel R. W. An extended Pontryagin principle for control systems whose control laws contain measures. *J. Soc. Indust. Appl. Math. Ser. A Control*, 1965, vol. 3, pp. 191–205. doi: 10.1137/0303016 .
15. Warga J. Optimal control of differential and functional equations. N Y; London: Acad. Press, 1972, 531 p. ISBN: 0127351507 .
16. Zavalishchin S. T., Sesekin A. N. *Dynamic impulse systems. Theory and applications.* Dordrecht: Kluwer Acad. Publ. Group, 1997, Ser. Math. Appl., vol. 394, 256 p. doi: 10.1007/978-94-015-8893-5 .
17. Brockett R. Notes on the control of the Liouville equation. In: P. Cannarsa, J.-M. Coron (eds.), *Control of partial differential equations.* Heidelberg: Springer, 2012, Ser. Lecture Notes in Math., vol. 2048. pp. 101–129. doi: 10.1007/978-3-642-27893-8_2 .
18. Ambrosio L., Savaré G. Gradient flows of probability measures. In C.M. Dafermos, E. Feireisl (eds.), *Handbook of differential equations: evolutionary equations*, Vol. III, Amsterdam: Elsevier, 2007, pp. 1–136. ISBN: 978-0-444-52848-3 .
19. Santambrogio F. *Optimal transport for applied mathematicians. Calculus of variations, PDEs, and modeling.* Cham: Birkhäuser Springer, 2015, 353 p. doi: 10.1007/978-3-319-20828-2 .
20. Bonnet B., Rossi F. The Pontryagin Maximum Principle in the Wasserstein space. *Calc. Var. Partial Differ. Equ.*, 2019, vol. 58, article 11, 36 p. doi: 10.1007/s00526-018-1447-2 .
21. Averboukh Y. Viability theorem for deterministic mean field type control systems. *Set-Valued and Variational Analysis*, 2018, vol. 26, no. 4, pp. 993–1008. doi: 10.1007/s11228-018-0479-2 .
22. Staritsyn M. On "discontinuous" continuity equation and impulsive ensemble control. *Systems and Control Letters*, 2018, vol. 118, pp. 77–83. doi: 10.1016/j.sysconle.2018.06.001 .
23. Staritsyn M.V., Pogodaev N.I. On a class of impulsive control problems for continuity equations. *IFAC-PapersOnLine*, (17th IFAC Workshop on Control Applications of Optimization CAO 2018) 2018, vol. 51, no. 32, pp. 468–473 doi: 10.1016/j.ifacol.2018.11.429 .
24. Pogodaev N. Optimal control of continuity equations. *NoDEA Nonlinear Differential Equations Appl.*, 2016, vol. 23, article 21, 24 p. doi: 10.1007/s00030-016-0357-2 .
25. Piccoli B., Rossi F. Transport equation with nonlocal velocity in Wasserstein spaces: convergence of numerical schemes., *Acta Appl. Math.*, 2013, vol. 124, no. 1, pp. 73–105. doi: 10.1007/s10440-012-9771-6 .
26. Pogodaev N., Staritsyn M. Impulsive relaxation of continuity equations and modeling of colliding ensembles. In: Evtushenko Y. et al (eds.), *Optimization and Applications.* Communications in Computer and Information Science, vol. 974. 2019, pp. 367–381. doi: 10.1007/978-3-030-10934-9_26 .
27. Ovsyannikov D.A. *Matematicheskie metody upravleniya puchkami* [Mathematical methods of beam control]. Leningrag: Leningrad State Univer. Publ., 1980, 228 p.
28. Pogodaev N. Program strategies for a dynamic game in the space of measures. *Optim. Lett.*, 2018, pp. 1–13. doi: 10.1007/s11590-018-1318-y .
29. Kipka R.J., Ledyaev Y.S. Extension of chronological calculus for dynamical systems on manifolds. *J. Diff. Eq.*, 2015, vol. 258, no. 5, pp. 1765–1790. doi: 10.1016/j.jde.2014.11.014 .
30. Filippov A.F. On some questions in the theory of optimal regulation: existence of a solution of the problem of optimal regulation in the class of bounded measurable functions. *Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him.*, 1959, no. 2, pp. 25–32 (in Russian).
31. Ioffe A.D., Tihomirov V.M. *Theory of extremal problems.* N Y, Oxford, North-Holland Publ. Comp., 1979, 460 p. ISBN: 0444851674 . Original Russian text published in Ioffe A.D., Tihomirov V.M. *Teoriya ekstremal'nykh zadach.* Moscow, Nauka Publ., 1974, 481 p.

Received December 15, 2018
Revised February 5, 2019
Accepted February 11, 2019

Funding Agency: This work was supported by the Russian Foundation for Basic Research (projects no. 18-31-20030, no. 18-01-00026).

Maksim Vladimirovich Staritsyn, Cand. Sci. (Phys.-Math.), Matrosov Institute for System Dynamics and Conrol Theory of the Siberian Branch of the Russian Academy of Sciences, Irkutsk, 664033 Russia,
e-mail: starmaxmath@gmail.com .

Nikolay Il'ich Pogodaev, Cand. Sci. (Phys.-Math.), Matrosov Institute for System Dynamics and Conrol Theory of the Siberian Branch of the Russian Academy of Sciences, Irkutsk, 664033 Russia,
e-mail: nickpogo@gmail.com .

Cite this article as:
M. V. Staritsyn, N. I. Pogodaev. On a class of problems of optimal impulse control for a continuity equation, *Trudy Instituta Matematiki i Mekhaniki URO RAN*, 2019, vol. 25, no. 1, pp. 229–244.