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# NIKOLSKII – BERNSTEIN CONSTANTS FOR NONNEGATIVE ENTIRE FUNCTIONS OF EXPONENTIAL TYPE ON THE AXIS

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We investigate a weighted version of the Nikolskii–Bernstein inequality

$$\|\Lambda_{\alpha}^{k} f\|_{q,\alpha} \leq \mathcal{L}(\alpha,p,q,k)\sigma^{(2\alpha+2)(1/p-1/q)+k}\|f\|_{p,\alpha}, \quad \alpha \geq -1/2,$$

on the subspace $\mathcal{E}^{\sigma} \cap L^{p}(\mathbb{R},|x|^{2\alpha+1}\,dx)$ of entire functions of exponential type. Here $\Lambda_{\alpha}$ is the Dunkl differential-difference operator whose second power generates the Bessel differential-difference operator $B_{\alpha}$. For $(p,q)=(1,\infty)$, we compute the following sharp constants for nonnegative functions:

$$\mathcal{L}_0^*(\alpha)_+ = \frac{1}{2^{2\alpha+2}}, \quad \mathcal{L}_1^*(\alpha)_+ = \frac{1}{2^{2\alpha+4}(\alpha+2)},$$

where $\mathcal{L}_r^*(\alpha)_+ = (\alpha+1)c_{\alpha}^{-2}\mathcal{L}(\alpha,1,\infty,2r)_+$ denotes the normalized Nikolskii–Bernstein constant. There are unique (up to a constant factor) extremizers $j_{\alpha+1}^2(x/2)$ and $x^2 j_{\alpha+2}^2(x/2)$, respectively. These results are proved with the use of the Markov quadrature formula with nodes at zeros of the Bessel function and the following generalization of Arestov, Babenko, Deikalova, and Horváth's recent result:

$$\mathcal{L}(\alpha,p,\infty,2r) = \sup B_{\alpha}^{r} f(0), \quad r \in \mathbb{Z}_+,$$

where the supremum is taken over all even real functions on $\mathbb{R}$ belonging to $\mathcal{E}_{p,\alpha}^{1}$. Our approach is based on the one-dimensional Dunkl harmonic analysis. In particular, we use the even positive Dunkl-type generalized translation operator $T_{\alpha}^{t}$, which is bounded on $L^{p}(\mathbb{R},|t|^{2\alpha+1}\,dt)$ with constant 1, is invariant on the subspace $\mathcal{E}_{p,\alpha}^{\sigma}$, and commutes with $B_{\alpha}$.

Keywords: weighted Nikolskii–Bernstein inequality, sharp constant, entire function of exponential type, Dunkl transform, generalized translation operator, Bessel function.

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