BEST UNIFORM APPROXIMATION OF THE DIFFERENTIATION OPERATOR BY OPERATORS BOUNDED IN THE SPACE L_2

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We give a solution of the problem on the best uniform approximation on the numerical axis of the first-order differentiation operator on the class of functions with bounded second derivative by linear operators bounded in the space L_2. This is one of the few cases of the exact solution of the problem on the approximation of the differentiation operator in some space with the use of approximating operators that are bounded in another space. We obtain a related exact inequality between the uniform norm of the derivative of a function, the variation of the Fourier transform of the function, and the L_∞-norm of its second derivative. This inequality can be regarded as a nonclassical variant of the Hadamard–Kolmogorov inequality.

Keywords: Stechkin problem, differentiation operator, Hadamard–Kolmogorov inequality.

REFERENCES

1. Stechkin S.B. Best approximation of linear operators. *Math. Notes*, 1967, vol. 1, no. 2, pp. 91–99. doi: 10.1007/BF01268056 .

2. Arestov V.V. Approximation of unbounded operators by bounded operators and related extremal problems. *Russian Math. Surveys*, 1996, vol. 51, no. 6, pp. 1093–1126. doi: 10.1070/RM1996v051n06ABEH003001 .

3. Babenko V.F., Korneichuk N.P., Kofanov V.A., Pichugov S.A. *Neravenstva dlya proizvodnykh i ikh prilozheniya* [Inequalities for derivatives and their applications]. Kiev, Naukova Dumka Publ., 2003, 591 p. (in Russian).

4. Ivanov V.K., Vasin V.V., Tanana V.P. *Theory of linear ill-posed problems and its applications.* Utrecht: VSP, 2002, 294 p. ISBN: 978-3111826141 . Original Russian text published in V.K. Ivanov, V.V. Vasin, V.P. Tanana, *Teoriya lineinykh nekorrektnykh zadach i ee prilozheniya.* Moscow: Nauka Publ., 1978, 206 p.

5. Gabushin V.N. Best approximations of functionals on certain sets, *Math. Notes*, 1970, vol. 8, no. 5, pp. 780–785. doi: 10.1007/BF01146932 .

6. Babenko Yu., Skorokhodov D. Stechkin's Problem for differential operators and functionals of first and second orders. *J. Approx. Theory*, 2013, vol. 167, pp. 173–200. doi: 10.1016/j.jat.2012.12.003 .

7. Babenko V.F., Parfinovich N.V., Pichugov S.A. Kolmogorov-type inequalities for norms of Riesz derivatives of functions of several variables with Laplacian bounded in L_∞ and related problems. *Math. Notes*, 2014, vol. 95, no. 1, pp. 3–14. doi: 10.1134/S0001434614010015 .

8. Berdysheva E., Filatova M. On the best approximation of the infinitesimal generator of a contraction semigroup in a Hilbert space. *Ural Math. J.*, 2017, vol. 3, no. 2, pp. 40–45. doi: 10.15826/umj.2017.2.006 .

9. Akopyan R.R. Approximation of the differentiation operator on the class of functions analytic in an annulus. *Ural Math. J.*, 2017, vol. 3, no. 2, pp. 6–13. doi: 10.15826/umj.2017.2.002 .

10. Akopyan R.R. Optimal recovery of a derivative of an analytic function from values of the function given with an error on a part of the boundary. *Analysis Math.*, 2018, vol. 44, no. 1, pp. 3–19. doi: 10.1007/s10476-018-0102-7 .

11. Arestov V.V. On the best approximation of the differentiation operator. *Ural Math. J.*, 2015, vol. 1, no. 1, pp. 20–29. doi: 10.15826/umj.2015.1.002 .

12. Arestov V.V., Filatova M.A. Best approximation of the differentiation operator in the space L_2 on the semiaxis. *J. Approx. Theory*, 2014, vol. 187, pp. 65–81. doi: 10.1016/j.jat.2014.08.001 .

13. Buslaev A.P., Magaril-Il'yaev G.G., Tikhomirov V.M. Existence of extremal functions in inequalities for derivatives. *Math. Notes*, 1982, vol. 32, no. 6, pp. 898–904. doi: 10.1007/BF01145874 .

14. Stein E.M., Weiss G. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton: Princeton Univ. Press, 1971, 312 p. ISBN: 9781400883899 . Translated to Russian under the title *Vvedenie v garmonicheskii analiz na evklidovykh prostranstvakh.* Moscow, Mir Publ., 1974, 333 p.

15. Landau E. Einige Ungleichungen für zweimal differentierbare Funktionen. *Proc. London Math. Soc. (2)*, 1913, vol. 13, pp. 43–49. doi: 10.1112/plms/s2-13.1.43 .

16. Kolmogorov A.N. On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval. *Selected works. Mathematics and Mechanics.* Moscow: Nauka Publ., 1985, pp. 252–263. (*Moskov. Gos. Univ., Uchenye Zap. (Mat. 3)*, 1939, vol. 30, pp. 3–16) (in Russian).

17. Hadamard J. Sur le module maximum d'une fonction et de ses dérivées. *Soc. math. France, Comptes rendus des Séances*, 1914, vol. 41, pp. 68–72.

18. Bosse Yu.G. (Shilov G.E.) On inequalities between derivatives. *Collection of Works of Student Scientific Societies of Moscow State University*, 1937, vol. 1, pp. 17–27 (in Russian).

19. Gabushin V.N. The best approximation of the differentiation operator in the metric of L_p. *Math. Notes*, 1972, vol. 12, no. 5, pp. 756–760. doi: 10.1007/BF01099059 .

20. Tikhomirov V.M., Magaril-Il'yaev G.G. Inequalities for derivatives. In: *Kolmogorov A.N. Selected Works*: *Mathematics and Mechanics*, Moscow, Nauka Publ., 1985, pp. 387–390 (in Russian).

21. Arestov V.V. Approximation of operators invariant with respect to a shift. *Proc. Steklov Inst. Math.*, 1975, vol. 138, pp. 45–74.

22. Arestov V.V. Approximation of operators of convolution type by bounded linear operators. *Proc. Steklov Inst. Math.*, 1981, vol. 145, pp. 1–18.

23. Arestov V.V. Approximation of invariant operators. *Math. Notes*, 1983, vol. 34, no. 1, pp. 489–499. doi: 10.1007/BF01160861 .

24. Arestov V.V. On the best approximation of the differentiation operator. In: *Approximation of functions by polynomials and splines, collected papers*, Sverdlovsk, 1985, pp. 3–14 (in Russian).

25. Arestov V.V. Best approximation of unbounded shift-invariant operators by linear bounded operators. *Proc. Steklov. Inst. Math.*, 1994, vol. 198, pp. 1–16.

26. Hörmander L. Estimates for translation invariant operators in L^p spaces. *Acta Mathematica*, 1960, vol. 104, no. 1-2, pp. 93–140. doi: 10.1007/BF02547187 .

27. Larsen R. *An introduction to the theory of multipliers.* Berlin etc.: Springer, 1971, 282 p. doi: 10.1007/978-3-642-65030-7 .

28. Stechkin S.B. Inequalities between norms of derivatives of arbitrary functions. *Acta Sci. Math.*, 1965, vol. 26, no. 3-4, pp. 225–230 (in Russian).

29. Arestov V.V. On the best approximation of differentiation operators. *Math. Notes*, 1967, vol. 1, no. 2, pp. 100–103. doi: 10.1007/BF01268057 .

30. Buslaev A.P. Approximation of a differentiation operator. *Math. Notes*, 1981, vol. 29, no. 5, pp. 372–378. doi: 10.1007/BF01158361 .

31. Domar Y. An extremal problem related to Kolmogoroff's inequality for bounded functions. *Arkiv för Mat.*, 1968, vol. 7, no. 5, pp. 433–441. doi: 10.1007/BF02590991 .

32. Subbotin Yu.N., Taikov L.V. Best approximation of a differentiation operator in L_2-space, *Math. Notes*, 1968, vol. 3, no. 2, pp. 100–105. doi: 10.1007/BF01094328 .

33. Hardy G.H., Littlewood J.E., Pólya G. *Inequalities.* Cambridge: Cambridge Univ. Press, 1934, 314 p. ISBN(2nd ed.): 0-521-05206-8 . Translated to Russian under the title *Neravenstva.* Moscow: Inostr. Lit. Publ., 1948, 456 p.

34. Gradshteyn I.S., Ryzhik I.M. *Table of integrals, series, and products.* N Y; London; Oxford: Elsevier, Acad. Press, 2007, 1172 p. ISBN: 978-0-12-373637-6 . Translated to Russian under the title *Tablitsy integralov, summ, ryadov i proizvedenii.* 2011, Saint Petersburg: BKhV-Peterburg, 2011, 1232 p.

35. Pólya G., Szegő G. *Problems and theorems in analysis.* Vol. 2. Berlin: Springer, 1972, 392 p. ISBN: 978-3-540-63686-1 . Translated to Russian under the title *Zadachi i teoremy iz analiza.* T. 2. Moscow: Nauka Publ., 1978, 432 p.

36. Markushevich A.I. *Teoriya analiticheskikh funktsii* [Theory of analytic functions]. Vol. 1. Moscow: Nauka Publ., 1967, 488 p. ISBN(3rd ed.): 978-5-8114-0928-0 .

37. Fikhtengol'ts G.M. *Kurs differentsial'nogo i integral'nogo ischisleniya* [A course in differential and integral calculus]. Vol. 2. St.-Petersburg: Lan' Publ., 1997, 800 p. ISBN(10th ed.): 978-5-8114-0674-6.

38. Dunford N., Schwartz J.T. *Linear operators. Part* 1: *General theory.* N Y: Interscience, 1988. 872 p. ISBN: 978-0-471-60848-6 . Translated to Russian under the title *Lineinye operatory. Obshchaya teoriya.* Moscow: Editorial URSS, 2004, 896 p.

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