TRUDY INSTITUTA MATEMATIKI I MEKHANIKI UrO RAN
Vol. 24 No. 4 **2018**

MSC: 42C10, 47A58, 30E10, 32E05
DOI: 10.21538/0134-4889-2018-24-4-270-282

On KOLMOGOROV TYPE INEQUALITIES IN THE BERGMAN SPACE FOR FUNCTIONS OF TWO VARIABLES

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Suppose that $\mathbf{z} := (\xi,\zeta) = (re^{it}, \rho e^{i\tau})$, where $0 \le r,\rho < \infty$ and $0 \le t,\tau \le 2\pi$, is a point in the two-dimensional complex space $\mathbb{C}^2$; $U^2 := \{\mathbf{z} \in \mathbb{C}^2 : |\xi| < 1,\, |\zeta| < 1\}$ is the unit bidisk in $\mathbb{C}^2$; $\mathcal{A}(U^2)$ is the class of functions analytic in U^2; and $B_2 := B_2(U^2)$ is the Bergman space of functions $f \in \mathcal{A}(U^2)$ such that

$$\|f\|_2 := \|f\|_{B_2(U^2)} = \left(\frac{1}{4\pi^2}\iint_{(U^2)} |f(\xi,\zeta)|^2 d\sigma_\xi d\sigma_\zeta\right)^{1/2} < +\infty,$$

where $d\sigma_\xi := dxdy$, $d\sigma_\zeta := dudv$, and the integral is understood in the Lebesgue sense. S. B. Vakarchuk and M. B. Vakarchuk (2013) proved that, under some conditions on the Taylor coefficients $c_{pq}(f)$ in the expansion of $f(\xi,\zeta)$ in a double Taylor series, the following exact Kolmogorov inequality holds:

$$\left\|f^{(k-\mu,l-\nu)}\right\|_2 \le \mathcal{C}_{k,l}(\mu,\nu)\,\|f\|_2^{\mu\nu/(kl)}\,\left\|f^{(k,0)}\right\|_2^{(1-\mu/k)\nu/l}\,\left\|f^{(0,l)}\right\|_2^{(1-\nu/l)\mu/k}\,\left\|f^{(k,l)}\right\|_2^{(1-\mu/k)(1-\nu/l)},$$

where the numerical coefficients $\mathcal{C}_{k,l}(\mu,\nu)$ are explicitly defined by the parameters $k,l \in \mathbb{N}$ and $\mu,\nu \in \mathbb{Z}_+$. We find an exact Kolmogorov type inequality for the best approximations $\mathscr{E}_{m-1,n-1}(f)_2$ of functions $f \in B_2(U^2)$ by generalized polynomials (quasipolynomials):

$$\mathscr{E}_{m-k+\mu-1,n-l+\nu-1}\big(f^{(k-\mu,l-\nu)}\big)_2$$

$$\le \frac{\alpha_{m,k-\mu}\alpha_{n,l-\nu}(m-k+1)^{(k-\mu)/(2k)}(n-l+1)^{(l-\nu)/(2l)}(m+1)^{\mu/(2k)}(n+1)^{\nu/(2l)}}{(\alpha_{m,k})^{1-\mu/m}(\alpha_{n,l})^{1-\nu/l}\,[(m-k+\mu+1)(n-l+\nu+1)]^{1/2}}$$

$$\times \big(\mathscr{E}_{m-1,n-1}(f)_2\big)^{\frac{\mu\nu}{kl}}\big(\mathscr{E}_{m-k-1,n-l}\big(f^{(k,0)}\big)_2\big)^{(1-\frac{\mu}{k})\frac{\nu}{l}}$$

$$\times \big(\mathscr{E}_{m-1,n-l-1}\big(f^{(0,l)}\big)_2\big)^{\frac{\mu}{k}(1-\frac{\nu}{l})}\big(\mathscr{E}_{m-k-1,n-l-1}\big(f^{(k,l)}\big)_2\big)^{(1-\frac{\mu}{k})(1-\frac{\nu}{l})}$$

in the sense that there exists a function $f_0 \in B_2^{(k,l)}$ for which the inequality turns into an equality.

Keywords: Kolmogorov type inequality, Bergman space, analytic function, quasipolynomial, upper bound.

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Received July 3, 2018
Revised October 19, 2018
Accepted October 22, 2018

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Cite this article as:
M. Sh. Shabozov, V. D. Sainakov. On Kolmogorov type inequalities in the Bergman space for functions of two variables, *Trudy Inst. Mat. Mekh. UrO RAN*, 2018, vol. 24, no. 4, pp. 270–282 .