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BERNSTEIN–SZEGŐ INEQUALITY FOR THE WEYL DERIVATIVE OF TRIGONOMETRIC POLYNOMIALS IN L_0

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In the set $\mathscr{T}_n$ of trigonometric polynomials f_n of order n with complex coefficients, we consider Weyl (fractional) derivatives $f_n^{(\alpha)}$ of real nonnegative order α. The inequality $\left\|D_\theta^\alpha f_n\right\|_p \le B_n(\alpha,\theta)_p\|f_n\|_p$ for the Weyl–Szegő operator $D_\theta^\alpha f_n(t) = f_n^{(\alpha)}(t)\cos\theta + \tilde{f}_n^{(\alpha)}(t)\sin\theta$ in the set $\mathscr{T}_n$ of trigonometric polynomials is a generalization of the Bernstein inequality. Such inequalities have been studied for 90 years. G. Szegő obtained the exact inequality $\left\|f_n'\cos\theta + \tilde{f}_n'\sin\theta\right\|_\infty \le n\,\|f_n\|_\infty$ in 1928. Later on, A. Zygmund (1933) and A. I. Kozko (1998) showed that, for $p \ge 1$ and real $\alpha \ge 1$, the constant $B_n(\alpha,\theta)_p$ is equal to n^α for all $\theta \in \mathbb{R}$. The case $p = 0$ is of additional interest because it is in this case that $B_n(\alpha,\theta)_p$ is largest over $p \in [0,\infty]$. In 1994 V. V. Arestov (1994) showed that, for $\theta = \pi/2$ (in the case of the conjugate polynomial) and integer nonnegative α, the quantity $B_n(\alpha,\pi/2)_0$ grows exponentially in n as $4^{n+o(n)}$. It follows from his result that the behavior of the constant for $\theta \ne 2\pi k$ is the same. However, in the case $\theta = 2\pi k$ and $\alpha \in \mathbb{N}$, Arestov showed in 1979 that the exact constant is n^α. The author investigated the Bernstein inequality in the case $p = 0$ for positive noninteger α earlier (2018). The logarithmic asymptotics of the exact constant was obtained: $\sqrt[n]{B_n(\alpha,0)_0} \to 4$ as $n \to \infty$. In the present paper, this result is generalized to all $\theta \in \mathbb{R}$.

Keywords: trigonometric polynomial, Weyl derivative, conjugate polynomial, Bernstein–Szegő inequality, space L_0.

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