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ON THE EQUIVALENCE OF SOME RELATIONS IN DIFFERENT METRICS BETWEEN NORMS, BEST APPROXIMATIONS, AND MODULI OF SMOOTHNESS OF PERIODIC FUNCTIONS AND THEIR DERIVATIVES

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We propose a method capable, in particular, of establishing the equivalence of known upper estimates for the $L_q(\mathbb{T})$-norm $\|f^{(r)}\|_q$, the best approximation $E_{n-1}(f^{(r)})_q$, and the kth-order modulus of smoothness $\omega_k(f^{(r)};\pi/n)_q$ in terms of elements of the sequence $\{E_{n-1}(f)_p\}_{n=1}^{\infty}$ of best approximations of a 2π-periodic function $f\in L_p(\mathbb{T})$ by trigonometric polynomials of order at most $n-1$, $n\in\mathbb{N}$, where $r\in\mathbb{Z}_+$ $(f^{(0)}=f)$, $1<p<q<\infty$, and $\mathbb{T}=(-\pi,\pi]$. The principal result of the paper is the following statement. Let $1<p<q<\infty$, $r\in\mathbb{Z}_+$, $k\in\mathbb{N}$, $\sigma=r+1/p-1/q$, $f\in L_p(\mathbb{T})$, and $E(f;p;\sigma;q)\equiv\Big(\sum_{\nu=1}^{\infty}\nu^{q\sigma-1}E_{\nu-1}^q(f)_p\Big)^{1/q}<\infty$. Then the following inequalities are equivalent in the sense that each of them implies the other two:

(a) $\|f^{(r)}\|_q\le C_1(r,p,q)\,\{(1-\chi(r))\|f\|_p+E(f;p;\sigma;q)\}$;

(b) $E_{n-1}(f^{(r)})_q\le C_2(r,p,q)\left\{n^{\sigma}E_{n-1}(f)_p+\Big(\sum_{\nu=n+1}^{\infty}\nu^{q\sigma-1}E_{\nu-1}^q(f)_p\Big)^{1/q}\right\}$, $n\in\mathbb{N}$;

(c) $\omega_k(f^{(r)};\pi/n)_q\le C_3(k,r,p,q)\Big\{\Big(\sum_{\nu=n+1}^{\infty}\nu^{q\sigma-1}E_{\nu-1}^q(f)_p\Big)^{1/q}+n^{-k}\Big(\sum_{\nu=1}^{n}\nu^{q(k+\sigma)-1}E_{\nu-1}^q(f)_p\Big)^{1/q}\Big\}$, $n\in\mathbb{N}$.

Inequalities (a), (b), and (c) depend on the key estimate

$$\big\|S_m^{(l)}(f;\cdot)\big\|_q\le C_4(l,p,q)\Big\{(1-\chi(l))\|f\|_p+\Big(\sum_{\nu=1}^{m}\nu^{q\lambda-1}E_{\nu-1}^q(f)_p\Big)^{1/q}\Big\},\quad m\in\mathbb{N},$$

where $S_m(f;x)$ is the partial sum of order $m\in\mathbb{N}$ of the Fourier series of a function $f\in L_p(\mathbb{T})$, $l\in\mathbb{Z}_+$, $\lambda=l+1/p-1/q$, $\chi(t)=0$ for $t\le 0$, and $\chi(t)=1$ for $t>0$, $t\in\mathbb{R}$. The latter estimate in the case $l=r$ and $\lambda=\sigma$ provides a necessary and sufficient condition for the fulfillment of inequality (a) under the condition $E(f;p;\sigma;q)<\infty$, which guarantees that $f\in L_q^{(r)}(\mathbb{T})$, where $L_q^{(r)}(\mathbb{T})$ is the class of functions $f\in L_q(\mathbb{T})$ with absolutely continuous $(r-1)$th derivative and $f^{(r)}\in L_q(\mathbb{T})$. Necessary and sufficient conditions for the validity of inequalities (b) and (c) are also provided in terms of the behavior of elements of the sequence $\{\|S_m^{(l)}(f;\cdot)\|_q\}_{m=1}^{\infty}$.

Keywords: best approximation, modulus of smoothness, inequalities in different metrics, equivalent inequalities.

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