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EQUIVALENCE OF THE EXISTENCE OF NONCONJUGATE AND NONISOMORPHIC HALL π-SUBGROUPS

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Let π be some set of primes. A subgroup H of a finite group G is called a Hall π-subgroup if any prime divisor of the order $|H|$ of the subgroup H belongs to π and the index $|G:H|$ is not a multiple of any number in π. The famous Hall theorem states that a solvable finite group always contains a Hall π-subgroup and any two Hall π-subgroups of such group are conjugate. The converse of the Hall theorem is also true: for any nonsolvable group G, there exists a set π such that G does not contain Hall π-subgroups. Nevertheless, Hall π-subgroups may exist in a nonsolvable group. There are examples of sets π such that, in any finite group containing a Hall π-subgroup, all Hall π-subgroups are conjugate (and, as a consequence, are isomorphic). In 1987 F. Gross showed that any set π of odd primes has this property. In addition, in nonsolvable groups for some sets π, Hall π-subgroups can be nonconjugate but isomorphic (say, in $PSL_2(7)$ for $\pi=\{2,3\}$) and even nonisomorphic (in $PSL_2(11)$ for $\pi=\{2,3\}$). We prove that the existence of a finite group with nonconjugate Hall π-subgroups for a set π implies the existence of a group with nonisomorphic Hall π-subgroups. The converse statement is obvious.

Keywords: Hall π-subgroup, $\mathscr{C}_\pi$ condition, conjugate subgroups.

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