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STABILIZERS OF VERTICES OF GRAPHS WITH PRIMITIVE AUTOMORPHISM GROUPS AND A STRONG VERSION OF THE SIMS CONJECTURE. IV

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This is the fourth in a series of papers whose results imply the validity of a strengthened version of the Sims conjecture on finite primitive permutation groups. In this paper, the case of primitive groups with simple socle of orthogonal Lie type and nonparabolic point stabilizer is considered. Let G be a finite group, and let M_1 and M_2 be distinct conjugate maximal subgroups of G. For any $i \in \mathbb{N}$, we define inductively subgroups $(M_1, M_2)^i$ and $(M_2, M_1)^i$ of $M_1 \cap M_2$, which will be called the ith mutual cores of M_1 with respect to M_2 and of M_2 with respect to M_1, respectively. Put $(M_1, M_2)^1 = (M_1 \cap M_2)_{M_1}$ and $(M_2, M_1)^1 = (M_1 \cap M_2)_{M_2}$. For $i \in \mathbb{N}$, assuming that $(M_1, M_2)^i$ and $(M_2, M_1)^i$ are already defined, put $(M_1, M_2)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_1}$ and $(M_2, M_1)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_2}$. We are interested in the case when $(M_1)_G = (M_2)_G = 1$ and $1 < |(M_1, M_2)^2| \leq |(M_2, M_1)^2|$. The set of all such triples (G, M_1, M_2) is denoted by Π. We consider triples from Π up to the following equivalence: triples (G, M_1, M_2) and (G', M_1', M_2') from Π are equivalent if there exists an isomorphism from G to G' mapping M_1 to M_1' and M_2 to M_2'. In the present paper, the following theorem is proved.

Theorem. *Suppose that $(G, M_1, M_2) \in \Pi$, $L = Soc(G)$ is a simple orthogonal group of degree ≥ 7, and $M_1 \cap L$ is a nonparabolic subgroup of L. Then $L \cong O_8^+(r)$, where r is an odd prime, $(M_1, M_2)^3 = (M_2, M_1)^3 = 1$, and one of the following statements holds*:

(a) $r \equiv \pm 1 \pmod 8$*, the group G is isomorphic to $O_8^+(r) : \mathbb{Z}_3$ or $O_8^+(r) : S_3$, $(M_1, M_2)^2 = Z(O_2(M_1))$ and $(M_2, M_1)^2 = Z(O_2(M_2))$ are elementary abelian groups of order 2^3, $(M_1, M_2)^1 = O_2(M_1)$ and $(M_2, M_1)^1 = O_2(M_2)$ are special groups of order 2^9, the group $M_1/O_2(M_1)$ is isomorphic to $L_3(2) \times \mathbb{Z}_3$ or $L_3(2) \times S_3$, respectively, and $M_1 \cap M_2$ is a Sylow 2-subgroup in M_1*;

(b) $r \leq 5$; *G/L either contains $Outdiag(L)$ or is isomorphic to $\mathbb{Z}_4$, $(M_1, M_2)^2 = Z(O_2(M_1 \cap L))$ and $(M_2, M_1)^2 = Z(O_2(M_2 \cap L))$ are elementary abelian groups of order 2^2, $(M_1, M_2)^1 = [O_2(M_1 \cap L), O_2(M_1 \cap L)]$ and $(M_2, M_1)^1 = [O_2(M_2 \cap L), O_2(M_2 \cap L)]$ are elementary abelian groups of order 2^5, $O_2(M_1 \cap L)/[O_2(M_1 \cap L), O_2(M_1 \cap L)]$ is an elementary abelian group of order 2^6; the group $(M_1 \cap L)/O_2(M_1 \cap L)$ is isomorphic to the group S_3, $|M_1 : M_1 \cap M_2| = 24$, $|M_1 \cap M_2 \cap L| = 2^{11}$, and an element of order 3 from $M_1 \cap M_2$ (in the case $G/L \cong A_4$ or $G/L \cong S_4$) induces on the group L its standard graph automorphism.*

In any of cases (a) *and* (b)*, the triples (G, M_1, M_2) from Π exist and form one class up to equivalence.*

Keywords: finite primitive permutation group, stabilizer of a point, Sims conjecture, almost simple group, group of orthogonal Lie type.

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