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ON THE EQUIVALENCE OF SOME INEQUALITIES IN THE THEORY OF APPROXIMATION OF PERIODIC FUNCTIONS IN THE SPACES $L_P(\mathbb{T})$, $1 < P < \infty$

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We propose a method for proving, in particular, the equivalence of M. F. Timan's known estimates for the rth-order L_p-moduli of smoothness $\omega_r(f;\pi/n)_p$ and O. V. Besov's estimates for the L_p-norms $\|f^{(r)}\|_p$ of rth-order derivatives by using elements of the sequence $\{E_{n-1}(f)_p\}_{n=1}^{\infty}$ of the best approximations of a 2π-periodic function $f \in L_p(\mathbb{T})$ by trigonometric polynomials of order at most $n-1$, $n \in \mathbb{N}$, where $r \in \mathbb{N}$, $1 < p < \infty$, and $\mathbb{T} = (-\pi, \pi]$.

Theorem 1. *Let $1 < p < \infty$, $\theta = \min\{2, p\}$, $r \in \mathbb{N}$, $f \in L_p(\mathbb{T})$, and $\sum_{n=1}^{\infty} n^{\theta r-1} E_{n-1}^{\theta}(f)_p < \infty$. Then the inequality $\omega_r(f;\pi/n)_p \le C_1(r,p) n^{-r} \Big(\sum_{\nu=1}^{n} \nu^{\theta r-1} E_{\nu-1}^{\theta}(f)_p\Big)^{1/\theta}$, $n \in \mathbb{N}$, is satisfied if and only if $f \in L_p^{(r)}(\mathbb{T})$ and $\|f^{(r)}\|_p \le C_2(r,p)\Big(\sum_{n=1}^{\infty} n^{\theta r-1} E_{n-1}^{\theta}(f)_p\Big)^{1/\theta}$, where $L_p^{(r)}(\mathbb{T})$ is the class of functions $f \in L_p(\mathbb{T})$ with absolutely continuous derivative of the $(r-1)$th order and $f^{(r)} \in L_p(\mathbb{T})$.*

Theorem 2. *Suppose that $1 < p < \infty$, $\beta = \max\{2, p\}$, $r \in \mathbb{N}$, and $f \in L_p^{(r)}(\mathbb{T})$. Then the inequality $n^{-r}\Big(\sum_{\nu=1}^{n} \nu^{\beta r-1} E_{\nu-1}^{\beta}(f)_p\Big)^{1/\beta} \le C_3(r,p)\omega_r(f;\pi/n)_p$ is satisfied for $n \in \mathbb{N}$ if and only if the inequality $\Big(\sum_{n=1}^{\infty} n^{\beta r-1} E_{n-1}^{\beta}(f)_p\Big)^{1/\beta} \le C_4(r,p)\|f^{(r)}\|_p$ is satisfied.*

In view of the order identity $\sum_{\nu=1}^{n} \nu^{\alpha r-1} E_{\nu-1}^{\alpha}(f)_p \asymp \sum_{\nu=1}^{n} \nu^{\alpha r-1} \omega_l^{\alpha}(f;\pi/\nu)_p$, $n \in \mathbb{N} \cup \{+\infty\}$, where $1 \le \alpha < \infty$, $l \in \mathbb{N}$, and $l > r$, the assertions of Theorems 1 and 2 remain valid if we replace the sequence $\{E_{n-1}(f)_p\}_{n=1}^{\infty}$ by the sequence $\{\omega_l(f;\pi/n)_p\}_{n=1}^{\infty}$ (Theorems 3 and 4). The method used in the proof of Theorems 1 and 2 can be applied to derive equivalent upper estimates and equivalent lower estimates for the values $E_{n-1}(f^{(r)})_p$ and $\omega_k(f^{(r)};\pi/n)_p$, $n \in \mathbb{N}$, by means of elements of the sequence $\{E_{n-1}(f)_p\}_{n=1}^{\infty}$, where $k, r \in \mathbb{N}$ and $1 < p < \infty$.

Keywords: best approximation, modulus of smoothness, inequalities of approximation theory, equivalent inequalities, Timan's inequalities, Besov's inequalities.

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