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AN ESTIMATE FOR THE REMAINDER IN THE EXPANSION OF THE ELLIPTIC SINE

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The main result of the paper is a weighted estimate for the remainder $U_n(v,k)(1-k^2)^{n+1}$ of the asymptotic expansion of the elliptic sine $z=\mathrm{sn}(v;k^2)$ in powers of k^2-1 in the interval $[0,1)$. We show that

$$|(\cosh v)^2 U_n(v,k)(1-k^2)^{n+1}| \leqslant \mathrm{const}\,\frac{(1-k^2)^{n+1}}{(1-z)^{n+1}} \ (z\in[0,1),\ k\in[0,1)),$$

where the constant is independent of z and k. We also propose an algorithm for finding the terms of the expansion. The coefficients of the expansion can be formally obtained by the following scheme. We consider the Legendre elliptic integral of the first kind in the Jacobi form $v=u(z,k^2)$ $(z\in[0,1),\ k\in[0,1))$ and introduce an auxiliary function $v^{(0)}=u(z,1)$. At the first step, the function $z=\tanh v^{(0)}$ is expanded in a series in powers of $v-v^{(0)}$. Then we represent the difference $v-v^{(0)}=u(z,k^2)-u(z,1)$ as the Taylor series in powers of k^2-1 and substitute it into the expansion of the function $z=\tanh v^{(0)}$. In the coefficients of the expansion in powers of k^2-1, the variable z is replaced by the value $\tanh v^{(0)}$, which is expanded in powers of $v-v^{(0)}$. Next, the steps are repeated. Using this procedure, we can find all the coefficients of the asymptotic expansion of the elliptic sine $z=\mathrm{sn}(v;k^2)$ as $k\to 1$, although the procedure involves significant computational difficulties. The algorithm proposed in this paper is based on finding terms of the expansion that contribute to the remainder and estimating them.

Keywords: elliptic sine, asymptotic expansion, hyperbolic functions.

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