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AN ESTIMATE OF THE HAUSDORFF DISTANCE BETWEEN A SET AND ITS CONVEX HULL IN EUCLIDEAN SPACES OF SMALL DIMENSION

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We derive estimates for the Hausdorff distance between sets and their convex hulls in finite-dimensional Euclidean spaces with the standard scalar product and the corresponding norm. In the first part of the paper, we consider estimates for α-sets. By an α-set we mean an arbitrary compact set for which the parameter characterizing the degree of nonconvexity and computed in a certain way equals α. In most cases, the parameter α is the maximum possible angle under which the projections to this set are visible from points not belonging to the set. Note that α-sets were introduced by V. N. Ushakov for the classification of nonconvex sets according to the degree of their nonconvexity; α-sets are used for the description of wavefronts and for the solution of other problems in control theory. We consider α-sets only in a two-dimensional space. It is proved that, if α is small, then the corresponding α-sets are close to convex sets in the Hausdorff metric. This allows to neglect their nonconvexity and consider such sets convex if it is known that the parameter α is small. The known Shapley–Folkman theorem is often applied in the same way. In the second part of the paper we present some improvements of the estimates from the Shapley–Folkman theorem. The original Shapley–Folkman theorem states that the Minkowski sum of a large number of sets is close in the Hausdorff metric to the convex hull of this sum with respect to the value of the Chebyshev radius of the sum. We consider a particular case when the sum consists of identical terms; i.e., we add some set M to itself. For this case we derive an improved estimate, which is essential for sets in spaces of small dimension. In addition, as in Starr's known corollary, the new estimate admits the following improvement: the Chebyshev radius $R(M)$ on the right-hand side can be replaced by the inner radius $r(M)$ of the set M. However, as the dimension of the space grows, the new estimate tends asymptotically to the estimate following immediately from the Shapley–Folkman theorem.

Keywords: α-set, Minkowski sum, convex hull, Hausdorff distance.

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