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THE VOLUME OF A HYPERBOLIC TETRAHEDRON WITH SYMMETRY GROUP S_4

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The problem of calculating the volume of a hyperbolic tetrahedron of general form was solved in a number of works by G. Sforza and other authors. The formulas obtained are rather cumbersome. It is known that if a polyhedron has nontrivial symmetry, then the volume formula is essentially simplified. This phenomenon was discovered by Lobachevsky, who found the volume of an ideal tetrahedron. Later, J. Milnor expressed the corresponding volume as the sum of three Lobachevsky functions. In this paper we consider compact hyperbolic tetrahedra having the symmetry group S_4, which is generated by a mirror-rotational symmetry of the fourth order. The latter symmetry is the composition of rotation by the angle of $\pi/2$ about an axis passing through the middles of two opposite edges and reflection with respect to a plane perpendicular to this axis and passing through the middles of the remaining four edges. We establish necessary and sufficient conditions for the existence of such tetrahedra in $\mathbb{H}^3$. Then we find relations between their dihedral angles and edge lengths in the form of a cosine law. Finally, we obtain exact integral formulas expressing the hyperbolic volume of the tetrahedra in terms of the edge lengths.

Keywords: hyperbolic tetrahedron, symmetry group, reflection followed by a rotation, hyperbolic volume.

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