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ON THE ORDER OF DECREASE OF UNIFORM MODULI OF SMOOTHNESS FOR THE CLASSES OF PERIODIC FUNCTIONS $H_P^L[\omega]$, $l \in \ltimes$, $1 \le p < \infty$

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S. B. Stechkin posed the following problem: for given $1 \le p < q \le \infty$, $r \in \mathbb{Z}_+$, $l, k \in \mathbb{N}$, and $\omega \in \Omega_l(0,\pi]$, find the exact order of decrease of the $L_q(\mathbb{T})$-modulus of smoothness of the kth order $\omega_k(f^{(r)};\delta)_q$ on the classes of 2π-periodic functions $H_p^l[\omega] = \{f \in L_p(\mathbb{T}) : \omega_l(f;\delta)_p \le \omega(\delta),\ \delta \in (0,\pi]\}$, where $\mathbb{T} = (-\pi,\pi]$, $L_\infty(\mathbb{T}) \equiv C(\mathbb{T})$, and $\Omega_l(0,\pi]$ is the class of functions $\omega = \omega(\delta)$ defined on $(0,\pi]$ and satisfying the conditions $0 < \omega(\delta) \downarrow 0$ $(\delta \downarrow 0)$ and $\delta^{-l}\omega(\delta) \downarrow$ $(\delta \uparrow)$. Earlier the author solved this problem in the case $1 \le p < q < \infty$. In the present paper, we give a solution in the case $1 \le p < q = \infty$; more exactly, we prove the following theorems.

Theorem 1. *Suppose that* $1 \le p < \infty$, $f \in L_p(\mathbb{T})$, $r \in \mathbb{Z}_+$, $l, k \in \mathbb{N}$, $l > \sigma = r + 1/p$, $\rho = l - (k + \sigma)$, *and* $\sum_{n=1}^{\infty} n^{\sigma-1}\omega_l(f;\pi/n)_p < \infty$. *Then f is equivalent to some function* $\psi \in C^r(\mathbb{T})$ *and the following bound holds:* $\omega_k(\psi^{(r)};\pi/n)_\infty \le C_1(l,k,r,p)\Big\{\sum_{\nu=n+1}^{\infty}\nu^{\sigma-1}\omega_l(f;\pi/\nu)_p + \chi(\rho)n^{-k}\sum_{\nu=1}^{n}\nu^{k+\sigma-1}\omega_l(f;\pi/\nu)_p\Big\}$, $n \in \mathbb{N}$, *where* $\chi(t) = 0$ *for* $t \le 0$, $\chi(t) = 1$ *for* $t > 0$, *and* $C^r(\mathbb{T})$ *is the class of functions* $\psi \in C(\mathbb{T})$ *that have the usual* r*th-order derivative* $\psi^{(r)} \in C(\mathbb{T})$ *(we assume that* $\psi^{(0)} = \psi$ *and* $C^{(0)}(\mathbb{T}) = C(\mathbb{T})$*).*

Note that this bound covers all possible cases of relations between l and $k + r$.

Theorem 2. *Suppose that* $1 \le p < \infty$, $r \in \mathbb{Z}_+$, $l, k \in \mathbb{N}$, $l > \sigma = r + 1/p$, $\rho = l - (k + \sigma)$, $\omega \in \Omega_l(0,\pi]$, *and* $\sum_{n=1}^{\infty} n^{\sigma-1}\omega(\pi/n) < \infty$. *Then* $\sup\{\omega_k(\psi^{(r)};\pi/n)_\infty : f \in H_p^l[\omega]\} \asymp \sum_{\nu=n+1}^{\infty}\nu^{\sigma-1}\omega(\pi/\nu) + \chi(\rho)n^{-k} \times \sum_{\nu=1}^{n}\nu^{k+\sigma-1}\omega(\pi/\nu)$, $n \in \mathbb{N}$, *where ψ denotes the corresponding function from* $C^r(\mathbb{T})$ *equivalent to* $f \in H_p^l[\omega]$.

In Theorems 1 and 2, the case $l = k + \sigma = k + r + 1/p$ $(\Rightarrow \chi(\rho) = 0)$ is of the most interest. This case is possible only for $p = 1$, since $r \in \mathbb{Z}_+$ and $l, k \in \mathbb{N}$. In this case, the proof of the bound in Theorem 1 employs the inequality $n^{-l}\|T_{n,1}^{(l)}(f;\cdot)\|_\infty \le C_2(l)n\omega_{l+1}(f;\pi/n)_l$, where $T_{n,1}(f;\cdot)$ is a best approximation polynomial for the function $f \in L_1(\mathbb{T})$. The latter inequality is derived from the strengthened version of the inequality of different metrics for derivatives of arbitrary trigonometric polynomials $\|t_n^{(l)}(\cdot)\|_\infty \le 2^{-1}\pi\|t_n^{(l+1)}(\cdot)\|_1$, $n \in \mathbb{N}$.

Keywords: modulus of smoothness, best approximation, inequality between moduli of smoothness of different orders in different metrics, exact order of decrease for uniform moduli of smoothness on a class.

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