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STEINER'S PROBLEM IN THE GROMOV–HAUSDORFF SPACE: THE CASE OF FINITE METRIC SPACES

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We study Steiner's problem in the Gromov–Hausdorff space, i.e., in the space of compact metric spaces (considered up to isometry) endowed with the Gromov–Hausdorff distance. Since this space is not boundedly compact, the problem of the existence of a shortest network connecting a finite point set in this space is open. We prove that each finite family of finite metric spaces can be connected by a shortest network. Moreover, it turns out that there exists a shortest tree all of whose vertices are finite metric spaces. A bound for the number of points in such metric spaces is derived. As an example, the case of three-point metric spaces is considered. We also prove that the Gromov–Hausdorff space does not realise minimal fillings, i.e., shortest trees in it need not be minimal fillings of their boundaries.

Keywords: Steiner's problem, shortest network, Steiner's minimal tree, minimal filling, Gromov–Hausdorff space, Gromov–Hausdorff distance.

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