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EXPLICIT EXPRESSION FOR HYPERBOLIC LIMIT CYCLES OF A CLASS OF POLYNOMIAL DIFFERENTIAL SYSTEMS

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We consider systems of differential equations in the plane,

$$x' = \frac{dx}{dt} = P(x,y), \quad y' = \frac{dy}{dt} = Q(x,y),$$

where the dependent variables x and y and the independent one (the time) t are real, and $P(x,y)$, $Q(x,y)$ are polynomials in the variables x and y with real coefficients. These differential systems are mathematical models and arise in many fields of application like biology, economics, physics and engineering, etc. The existence of limit cycles is one of the more difficult objects to study in the qualitative theory of differential systems in the plane. There is a huge literature dedicated to this topic. It is known that for differential systems defined on the plane the existence of a first integral determines their phase portrait. Thus for polynomial differential systems a natural question arises: given a polynomial differential system in the plane, how to recognize if it has a first integral? There is a strong relation between the invariant algebraic curves and the theory of integrability. In this paper we introduce explicit expressions for invariant algebraic curves and for the first integral. Finally, we determine sufficient conditions for a class of polynomial differential systems to possess an explicitly given hyperbolic limit cycle. Concrete examples exhibiting the applicability of our results are introduced. The elementary method used in this paper seems to be fruitful to investigate more general planar dynamical systems in order to obtain explicitly some or all of their limit cycles at least in the case of hyperbolic cycles. In the spirit of the inverse approach to dynamical systems, we look for them as the ovals of suitably chosen invariant algebraic curves.

Keywords: planar polynomial differential system, invariant algebraic curve, first integral, limit cycle.

Р. Букуша. Явное выражение для гиперболических предельных циклов одного класса полиномиальных дифференциальных систем.

Рассматриваются системы дифференциальных уравнений на плоскости

$$x' = \frac{dx}{dt} = P(x,y), \quad y' = \frac{dy}{dt} = Q(x,y),$$

где зависимые переменные x и y, а также независимая переменная (время) t вещественны, а $P(x,y)$ и $Q(x,y)$ — вещественные многочлены от переменных x и y. Такие математические модели возникают во многих прикладных областях в биологии, экономики, технике и т.д. Существование предельных циклов представляет собой один из наиболее трудных для изучения вопросов качественной теории плоских дифференциальных систем, и этой теме посвящено огромное количество работ. Известно, что существование первого интеграла плоской дифференциальной системы определяет ее фазовый портрет. Таким образом, для полиномиальных дифференциальных систем возникает естественный вопрос: как определить, имеет ли данная система первый интеграл? Инвариантные алгебраические кривые тесно связаны с теорией интегрируемости. В данной статье введены явные выражения для инвариантных алгебраических кривых и для первого интеграла, а также найдены достаточные условия, при которых класс полиномиальных дифференциальных систем имеет явно заданные гиперболические предельные циклы. Приведены конкретные примеры, демонстрирующие применимость результатов. Представляется, что элементарный метод, использованный в данной статье, может быть применен для исследования более общих плоских динамических систем для получения в явном виде некоторых или всех предельных циклов, по крайней мере в случае гиперболических циклов. В духе обратного подхода к динамическим системам мы ищем их в виде овалов подходящих инвариантных алгебраических кривых.

Ключевые слова: плоская полиномиальная дифференциальная система, инвариантная алгебраическая кривая, первый интеграл, предельный цикл.

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Introduction

By definition, an autonomous planar polynomial system of differential equations is a system of the form

$$\begin{cases} x' = \dfrac{dx}{dt} = P(x,y), \\ y' = \dfrac{dy}{dt} = Q(x,y), \end{cases} \tag{0.1}$$

where P and Q are real polynomials in the variables x and y, we denote by $m = \max\{\deg P, \deg Q\}$ and we say that m is the degree of system (0.1). A limit cycle of system (0.1) is an isolated periodic solution in the set of all periodic solution of system (0.1) . In the qualitative theory of autonomous differential systems on the plane see [8;12], the study of limit cycles is very attractive because of their relation with the applications to other areas of sciences. This is the most important topics is related to the second part of the unsolved Hilbert 16th problem see [11]. There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly see [1–5; 10]. Another main open problems in the qualitative theory of real planar differential systems the determination of its first integrals, the importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals see [13]. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals, for or more details about first integral see for instance [6; 9] . It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait see [7].

1. Some useful notions

Let us recall some useful notions.

System (0.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (0.1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} P(x,y) + \frac{\partial H(x,y)}{\partial y} Q(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant.

Since for such vector fields the notion of integrability is based on the existence of a first integral, the following question arises: Given the polynomial differential systems (0.1), how to recognize if this polynomial differential systems has a first Integral? and how to compute it when it exists?

A curve $U(x,y) = 0$, where $U(x,y)$ is a polynomial with real coefficients, is an invariant algebraic curve of system (0.1) if and only if there exists a polynomial $K = K(x,y)$ of degree at most $m-1$ satisfying

$$\frac{\partial U(x,y)}{\partial x} P(x,y) + \frac{\partial U(x,y)}{\partial y} Q(x,y) = K(x,y)\, U(x,y). \tag{1.1}$$

The polynomial $K(x,y)$ is called the cofactor of $U(x,y) = 0$, if the cofactor is identically zero, then $U(x,y)$ is a polynomial first integral for system (0.1) . The corresponding cofactor of $U(x,y)$ is always polynomial whether $U(x,y)$ is algebraic or non algebraic. If U is real, the curve $U(x,y) = 0$ is an invariant under the flow of differential system (0.1) and the set $\left\{(x,y) \in \mathbb{R}^2,\ U(x,y) = 0\right\}$ is formed by orbits of system (0.1). There are strong relationships between the integrability of system (0.1) and its number of invariant algebraic solutions.

2. Main result

As a main result, we shall prove the following theorem.

Theorem. *Consider a multi-parameter planar polynomial differential system of the form*

$$\begin{cases} x' &= P_n(x,y), \\ y' &= Q_n(x,y), \end{cases} \tag{2.1}$$

where

$$P_n(x,y) = x + (y-x)\left(x^2 - xy + y^2\right)\left(ax^2 + bxy + ay^2\right)^n$$

and

$$Q_n(x,y) = y - (y+x)\left(x^2 - xy + y^2\right)\left(ax^2 + bxy + ay^2\right)^n,$$

in which a, b are real constants and $n \in \mathbb{N}$, then the following statements hold.

1) *The curve*

$$U(x,y) = -\left(x^2 - xy + y^2\right)\left(x^2 + y^2\right)\left(ax^2 + bxy + ay^2\right)^n$$

is an invariant algebraic curve of system (2.1) *with cofactor*

$$K(x,y) = 2n + 4 + \left(-3x^2 + 4xy - 5y^2\right)\left(ax^2 + bxy + ay^2\right)^n + n\Big((2ax + by)(y-x)\left(x^2 - xy + y^2\right) - (bx + 2ay)\left(x^3 + y^3\right)\Big)\left(ax^2 + bxy + ay^2\right)^{n-1}.$$

2) *If* $2a + b\sin 2\theta \neq 0$ *for all* $\theta \in \mathbb{R}$, *then system* (2.1) *has the first integral*

$$H(x,y) = \left(x^2 + y^2\right)^{n+1} \exp\left(-(2n+2)\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}} F(w)dw,$$

where

$$F(w) = \frac{(4n+4)\exp(-2nw - 2w)}{(-2 + \sin 2w)(a + 1/2\, b\sin 2w)^n}. \tag{2.2}$$

3) *If* $2a + b\sin 2\theta \neq 0$ *for all* $\theta \in \mathbb{R}$ *and* $2a + |b| > 0$, *then system* (2.1) *has limit cycle explicitly given in polar coordinates* (r, θ), *by*

$$r(\theta, r_*) = \exp(\theta)\left(r_*^{2n+2} + \int_0^{\theta} F(w)dw\right)^{1/(2n+2)},$$

where r_* a positive constant, such that

$$r_* = \left(\frac{\exp(4n\pi + 4\pi)}{1 - \exp(4n\pi + 4\pi)} \int_0^{2\pi} F(w)dw\right)^{1/(2n+2)}.$$

4) *If* $a = b = 0$, *then system* (2.1) *has the first integral* $H(x,y) = \dfrac{y}{x}$.
Moreover, the system (2.1) *has no limit cycle.*

Proof.

Proof of statement (1).

An computation shows that

$$U(x,y) = -\left(x^2 - xy + y^2\right)\left(x^2 + y^2\right)\left(ax^2 + bxy + ay^2\right)^n = 0,$$

satisfy the linear partial differential equation (1.1)

$$K(x,y) = 2n+4+\left(-3x^2+4xy-5y^2\right)\left(ax^2+bxy+ay^2\right)^n$$

$$+\,n\left(\left(2ax+by\right)\left(y-x\right)\left(x^2-xy+y^2\right)-\left(bx+2ay\right)\left(x^3+y^3\right)\right)\left(ax^2+bxy+ay^2\right)^{n-1},$$

then the curve $U(x,y)=0$ is an invariant algebraic curve of system (2.1) with cofactor $K(x,y)$.

This completes the proof of statement (1) of Theorem.

In order to prove our results (2)–(4) we write the polynomial differential system (2.1) in polar coordinates (r,θ), defined by $x=r\cos\theta$ and $y=r\sin\theta$, then system becomes

$$\begin{cases} r' = \dfrac{dr}{dt} = r+\Big(-1+\dfrac{1}{2}\sin 2\theta\Big)\Big(a+\dfrac{1}{2}b\sin 2\theta\Big)^n r^{2n+3}, \\ \theta' = \dfrac{d\theta}{dt} = \Big(-1+\dfrac{1}{2}\sin 2\theta\Big)\Big(a+\dfrac{1}{2}b\sin 2\theta\Big)^n r^{2n+2}. \end{cases} \tag{2.3}$$

Proof of statement (2).

If $2a+b\sin 2\theta\neq 0$ for all $\theta\in\mathbb{R}$, we take as new independent variable the variable θ, then the differential system (2.3) becomes

$$\frac{dr}{d\theta} = r+\left(\frac{2}{\left(-2+\sin 2\theta\right)\left(a+1/2\,b\sin 2\theta\right)^n}r\right)^{-2n-1}. \tag{2.4}$$

The equation (2.4) is a Bernoulli equation, by introducing the standard change of variables $\rho=r^{2n+2}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = \left(2n+2\right)\rho+\frac{4n+4}{\left(-2+\sin 2\theta\right)\left(a+1/2\,b\sin 2\theta\right)^n}. \tag{2.5}$$

The general solution of linear equation (2.5) is

$$\rho(\theta) = \exp\left(2n\theta+2\theta\right)\left(\lambda+\int_0^{\theta}F(w)dw\right),$$

where $\lambda\in(0,\infty)$ and $F(w)$ is defined in (2.2).

Then the general solution of Bernoulli equation (2.4) is

$$r(\theta) = \exp(\theta)\left(\lambda+\int_0^{\theta}F(w)dw\right)^{1/(2n+2)}, \tag{2.6}$$

where $\lambda\in(0,\infty)$, which has the first integral

$$H(x,y) = \left(x^2+y^2\right)^{n+1}\exp\Big(-\left(2n+2\right)\arctan\frac{y}{x}\Big)-\int_0^{\arctan\frac{y}{x}}F(w)dw.$$

Hence statement (2) of Theorem is proved.

Proof of statement (3).

If $2a+b\sin 2\theta\neq 0$ for all $\theta\in\mathbb{R}$, we have

$$yP_n(x,y)-xQ_n(x,y) = \left(x^2-xy+y^2\right)\left(x^2+y^2\right)\left(ax^2+bxy+ay^2\right)^n,$$

thus the origin is the unique critical point at finite distance.

It is easy to check that the solution $r(\theta, r_0)$ of the differential equation (2.6) such that $r(0, r_0) = r_0$ is

$$r(\theta, r_0) = \exp(\theta) \left(r_0^{2n+2} + \int_0^\theta F(w)dw \right)^{1/(2n+2)}. \tag{2.7}$$

A periodic solution of system (2.1) must satisfy the condition $r(2\pi, r_0) = r(0, r_0)$, which leads to an unique value $r_0 = r_*$, given by

$$r_* = \left(\frac{\exp(4n\pi + 4\pi)}{1 - \exp(4n\pi + 4\pi)} \int_0^{2\pi} F(w)dw \right)^{1/(2n+2)},$$

where $F(w)$ is defined in (2.2).

Since $2a + |b| > 0$, we have $a + 1/2\, b \sin 2w > 0$. Hence, $r_* > 0$.
Injecting this value of r_* in (2.7), we get the candidate solution

$$r(\theta, r_*) = \exp(\theta) \left(\frac{\exp(4n\pi + 4\pi)}{1 - \exp(4n\pi + 4\pi)} \int_0^{2\pi} F(w)dw + \int_0^\theta F(w)dw \right)^{1/(2n+2)}.$$

So, if $r(\theta, r_*) > 0$ for all $\theta \in \mathbb{R}$, we shall have $r(\theta, r_*)$ would be periodic orbit, and consequently a limit cycle.

Since $2a + |b| > 0$, we have $a + 1/2\, b \sin 2w > 0$, then $-F(w) > 0$ and

$$r(\theta, r_*) = \exp(\theta) \left(\frac{\exp(4n\pi + 4\pi)}{-1 + \exp(4n\pi + 4\pi)} \int_{2\pi}^{0} F(w)dw + \int_0^\theta F(w)dw \right)^{1/(2n+2)}$$

$$> \exp(\theta) \left(\int_\theta^{2\pi} -F(w)dw \right)^{1/(2n+2)} > 0,$$

for all $\theta \in (0, 2\pi)$.

Consequently, this is a limit cycle for the differential system (2.1).

In order to prove the hyperbolicity of the limit cycle it is sufficient to that the Poincaré return map, for more details see [8, section 1.6]. An computation shows that

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r_0 = r_*} = \exp(4n\pi + 4\pi) > 1,$$

Therefore the limit cycle of the differential equation (2.4) is hyperbolic. Consequently, this is a hyperbolic limit cycle for the differential system (2.1). This completes the proof of statement (3) of Theorem.

Proof of statement (4). Assume now that $a = b = 0$, then from (2.3) it follows that $\theta' = 0$. So the straight lines through the origin of coordinates of the differential system (2.1) are invariant by the flow of this system. Hence, $H(x, y) = y/x$ is a first integral of the system. This completes the proof of statement (4) of Theorem. □

3. Examples

The following examples is given to illustrate our results.

E x a m p l e 1. If we take $n = 0$, then system (2.1) reads

$$\begin{cases} x' = x + (y - x)\left(x^2 - xy + y^2\right), \\ y' = y - (y + x)\left(x^2 - xy + y^2\right). \end{cases} \tag{3.1}$$

The curve

$$U(x, y) \quad = \quad -\left(x^2 - xy + y^2\right)\left(x^2 + y^2\right) = 0,$$

is an invariant algebraic curve of system (3.1) with cofactor $K(x, y) = -3x^2 + 4xy - 5y^2 + 4$.

The system (3.1) has the first integral

$$H(x, y) = \left(x^2 + y^2\right)\exp\left(-2\arctan\frac{y}{x}\right) + \int_0^{\arctan\frac{y}{x}} \left(\frac{4\exp(-2w)}{2 - \sin 2w}\right) dw.$$

Moreover, the system (3.1) has limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \exp(\theta)\sqrt{r_*^2 - 4\int_0^{\theta}\left(\frac{\exp(-2w)}{2 - \sin 2w}\right) dw},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point having

$$r_* = \sqrt{\frac{2\exp(4\pi)}{\exp(4\pi) - 1}\int_0^{2\pi}\left(\frac{2}{2 - \sin 2w}\exp(-2w)\right) dw} \simeq 1.191\,2.$$

Moreover,

$$\left.\frac{dr(2\pi, r_0)}{dr_0}\right|_{r_0 = r_*} = e^{4\pi} > 1.$$

This limit cycle is hyperbolic limit cycle. It is the results presented by Jaume Llibre and Benterki Rebiha in [5].

E x a m p l e 2. If we take $n = 1$, $a = 3$ and $b = 2$ then system (2.1) reads

$$\begin{cases} x' = x + (y - x)\left(x^2 - xy + y^2\right)\left(3x^2 + 2xy + 3y^2\right), \\ y' = y - (y + x)\left(x^2 - xy + y^2\right)\left(3x^2 + 2xy + 3y^2\right). \end{cases} \tag{3.2}$$

The curve

$$U(x, y) = -\left(x^2 - xy + y^2\right)\left(x^2 + y^2\right)\left(3x^2 + 2xy + 3y^2\right) = 0,$$

is an invariant algebraic curve of system (3.2) with cofactor

$$K(x, y) = -17x^4 + 10x^3y - 24x^2y^2 + 2xy^3 - 19y^4 + 6.$$

The system (3.2) has the first integral

$$H(x, y) = \left(x^2 + y^2\right)^2\exp\left(-4\arctan\frac{y}{x}\right) - \int_0^{\arctan\frac{y}{x}}\left(\frac{8\exp(-4w)}{(-2 + \sin 2w)(3 + \sin 2w)}\right) dw.$$

Moreover, the system (3.2) has limit cycle whose expression in polar coordinates (r, θ) is

$$r(\theta, r_*) = \exp(\theta)\left(r_*^4 + \int_0^{\theta}\left(\frac{8\exp(-4w)}{(-2 + \sin 2w)(3 + \sin 2w)}\right) dw\right)^{1/4},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point having

$$r_* = \left(\frac{\exp 8\pi}{1-\exp 8\pi} \int\limits_0^{2\pi} \frac{8\exp(-4w)}{(-2+\sin 2w)(3+\sin 2w)} dw \right)^{1/4} \simeq 0.784\,63.$$

Moreover,

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r_0=r_*} = \exp(8\pi) > 1.$$

This limit cycle is hyperbolic limit cycle.

E x a m p l e 3. If we take $n = 2$, $a = 10$ and $b = -2$ then system (2.1) reads

$$\begin{cases} x' = x + (y-x)\left(x^2 - xy + y^2\right)\left(10x^2 - 2xy + 10y^2\right)^2, \\ y' = y - (y+x)\left(x^2 - xy + y^2\right)\left(10x^2 - 2xy + 10y^2\right)^2. \end{cases} \tag{3.3}$$

The curve

$$U(x,y) = -\left(x^2 - xy + y^2\right)\left(x^2 + y^2\right)\left(10x^2 - 2xy + 10y^2\right)^2 = 0,$$

is an invariant algebraic curve of system (3.3) with cofactor

$$K(x,y) = \left(-66x^4 + 90x^3y - 176x^2y^2 + 102xy^3 - 94y^4\right)\left(10x^2 - 2xy + 10y^2\right) + 8.$$

The system (3.3) has the first integral

$$H(x,y) = \left(x^2+y^2\right)^3 \exp\left(-6\arctan\frac{y}{x}\right) - \int\limits_0^{\arctan\frac{y}{x}} \left(\frac{12\exp(-6w)}{(-2+\sin 2w)(10-\sin 2w)^2} \right) dw.$$

Moreover, the system (3.3) has limit cycle whose expression in polar coordinates (r,θ) is

$$r(\theta, r_*) = \exp(\theta) \left(r_*^6 + \int\limits_0^{\theta} \left(\frac{12\exp(-6w)}{(-2+\sin 2w)(10-\sin 2w)^2} \right) dw \right)^{1/6},$$

where $\theta \in \mathbb{R}$, and the intersection of the limit cycle with the OX_+ axis is the point having

$$r_* = \left(\frac{\exp(12\pi)}{1-\exp(12\pi)} \int\limits_0^{2\pi} \frac{12\exp(-6w)}{(-2+\sin 2w)(10-\sin 2w)^2} dw \right)^{1/6} \simeq 0.484\,91.$$

Moreover,

$$\left. \frac{dr(2\pi, r_0)}{dr_0} \right|_{r_0=r_*} = \exp(12\pi) > 1.$$

This limit cycle is hyperbolic limit cycle.

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