Vol. 23 No. 3 **2017**

MSC: 41A10, 42A10

DOI: 10.21538/0134-4889-2017-23-3-43-57

MODIFIED BERNSTEIN FUNCTION AND A UNIFORM APPROXIMATION OF SOME RATIONAL FRACTIONS BY POLYNOMIALS

A. G. Babenko, Yu. V. Kryakin

P. L. Chebyshev posed and solved (1857, 1859) the problem of finding an improper rational fraction least deviating from zero in the uniform metric on a closed interval among rational fractions whose denominator is a fixed polynomial of a given degree m that is positive on the interval and numerator is a polynomial of a given degree $n \geq m$ with unit leading coefficient. A. A. Markov solved (1884) a similar problem in the case when the denominator is the square root of a given positive polynomial. In the 20th century, this research direction was developed by S. N. Bernstein, N. I. Akhiezer, and other mathematicians. For example, in 1964 G. Szegő extended Chebyshev's result to the case of trigonometric fractions using the methods of complex analysis. In this paper, using the methods of real analysis and developing Bernstein's approach, we find the best uniform approximation on a period by trigonometric polynomials of certain order for an infinite series of proper trigonometric fractions of a special form. It turned out that, in the periodic case, it is natural to formulate some results in terms of the generalized Poisson kernel $\Pi_{\rho,\xi}(t) = (\cos\xi)P_\rho(t) + (\sin\xi)Q_\rho(t)$, which is a linear combination of the Poisson kernel $P_\rho(t) = (1-\rho^2)/[2(1+\rho^2-2\rho\cos t)]$ and the conjugate Poisson kernel $Q_\rho(t) = \rho\sin t/(1+\rho^2-2\rho\cos t)$, where $\rho \in (-1,1)$ and $\xi \in \mathbb{R}$. We find the best uniform approximation on a period by the subspace $\mathcal{T}_n$ of trigonometric polynomials of order at most n for the linear combination $\Pi_{\rho,\xi}(t) + (-1)^n\Pi_{\rho,\xi}(t+\pi)$ of the generalized Poisson kernel and its shift. For $\xi = 0$, this yields Bernstein's known results on the best uniform approximation on $[-1,1]$ of the fractions $1/(x^2-a^2)$ and $x/(x^2-a^2)$ by algebraic polynomials. For $\xi = \pi/2$, we obtain the weight analogs (with weight $\sqrt{1-x^2}$) of these results. In addition, we find the value of the best uniform approximation on a period by the subspace $\mathcal{T}_n$ of a special linear combination of the mentioned Poisson kernel P_ρ and the Poisson kernel K_ρ for the biharmonic equation in the unit disk.

Keywords: Bernstein functions, Poisson kernels, uniform approximation.

REFERENCES

1. Akhiezer N. Sur la valeur asimptoticue de la meilleure approximation de quelques fractions par des polynômes. *Compt. Rend. Acad. Sci.*, 1930, vol. 191, pp. 991–993.
2. Achieser N.I. *Theory of approximation*, Reprint of the 1956, New York: Dover Publ., Inc., 1992, 307 p. ISBN: 0486671291 . Original Russian text published in *Lektsii po teorii approksimatsii.* Moscow, Leningrad: OGIZ Publ., 1947, 323 p.
3. Babenko A.G., Kryakin Yu.V. Integral approximation of the characteristic function of an interval by trigonometric polynomials. *Proc. Steklov Inst. Math.*, 2009, vol. 264, suppl. 1, pp. 19–38. doi: 10.1134/S0081543809050022 .
4. Babenko A.G., Kryakin V.Yu., Yudin V.A. On a result by geronimus. *Proc. Steklov Inst. Math.*, 2011, vol. 273, suppl. 1, pp. 37–48. doi: 10.1134/S008154381105004X .
5. Baraboshkina N.A. Approximation of harmonic functions by algebraic polynomials on a circle of radius smaller than one with constraints on the unit circle. *Trudy Inst. Mat. Mekh. UrO RAN*, vol. 19, no. 2, 2013, pp. 71–78 (in Russian).
6. Bernstein S.N. Collected Works (Russian): Vol. 1: The constructive theory of functions (1905–1930). U. S. Atomic Energy Commission, Springfield, Va, 1958, 221 p. Sobranie Sochinenii: Tom I. *Konstruktivnaya Teoriya Funktcii* (1905–1930). Transaltion of a publication of the Academy of Sciences of U.S.S.R. Press, 1952, Moscow.
7. Bernstein S.N. *Ekstremal'nye svoistva polinomov i nailuchshee priblizhenie nepreryvnykh funktsii odnoi veshchestvennoi peremennoi* [Extremal properties of polynomials and the best approximation of continuous functions of one real variable], Part 1, Moscow, Leningrad: ONTI NKTP SSSR Publ., 1937, 203 p.

8. Zygmund A. *Trigonometric series*, 2nd ed., New York: Cambridge University Press, 1959, vol. 1, 383 p. Translated under the title *Trigonometricheskie ryady*. Vol. 1, Moscow, Mir Publ., 1965, 616 p.
9. Dzjadyk V.K. On a problem of Chebyshev and Markov. *Analysis Math.*, 1977, vol. 3, pp. 171–175. doi: 10.1007/BF02297689 .
10. Lebedev V.I. Extremal polynomials and methods for the optimization of numerical algorithms. *Sb. Math.*, 2004, vol. 195, no. 9–10, pp. 1413–1459. doi: 10.1070/SM2004v195n10ABEH000852 .
11. Lukashov A.L. The algebraic fractions of Chebyshev and Markov on several segments. *Analysis Math.*, 1998, vol. 24, pp. 111–130 (in Russian). doi: 10.1007/BF02771077 .
12. Markov A.A. Determination of a function with respect to the condition to deviate as little as possible from zero. *Communication and Proceedings of the Mathematical Society of the Imperial University of Kharkov*, 1884, I, pp. 83–92.
13. Markov A.A. *Izbrannye trudy po teorii nepreryvnykh drobei i teorii funktsii, naimenee uklonyayushchikhsya ot nulya* [Selected works on the theory of continued fractions and the theory of functions least deviating from zero]. Moscow, Leningrad: Gostechizdat Publ., 1948, 411 p.
14. Nikol'skij S.M. *Kurs matematicheskogo analiza* [A course of mathematical analysis]. Moscow, Nauka Publ. , 1990, vol. 1, 528 p.
15. Paszkowski S. *Vychislitel'nye primenenija mnogochlenov i rjadov Chebyshjova* [Numerical applications of Chebyshev polynomials and series]. Transl. from Polish to Russian, Moscow, Nauka Publ., 1983, 384 p.
16. Rusak V.N. *Ratsional'nye funktsii kak apparat priblizheniya* [Rational functions as approximation apparatus]. Minsk, Beloruss. Gos. Univ. Publ., 1979, 176 p.
17. Szegö G. On a problem of the best approximation. *Abh. Math. Semin. Univ. Hamb.*, 1964, vol. 27, iss. 3, pp. 193–198. doi: 10.1007/BF02993216 .
18. Tikhonov A.N., Samarskii A.A. *Uravneniya matematicheskoi fiziki* [Equations of mathematical physics]. Moscow, Nauka Publ., 1977, 735 p.
19. Chebyshev P.L. *Complete Set of Works*. Moscow, Leningrad: Izd. Akad. Nauk SSSR, Vol. 2: Mathematical analysis, 1947, 520 p; Vol. 3: Mathematical analysis, 1948, 414 p. (in Russian).
20. Shabozov M.Sh. Best approximation and best unilateral approximation of the kernel of a biharmonic equation and optimal renewal of the values of operators, *Ukr. Math. J.* 1995, vol. 47, iss. 11, pp. 1769–1778.

The paper was received by the Editorial Office on October, 17, 2016.

Aleksandr Grigor'evich Babenko, Dr. Phys.-Math. Sci., Krasovskii Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Sciences, Yekaterinburg, 620990 Russia; Ural Federal University, Yekaterinburg, 620002 Russia, e-mail: babenko@imm.uran.ru .

Yuriy Kryakin, dr hab., Mathematical Institute of University of Wroclaw, 48-300 Wroclaw, Poland e-mail: kryakin@math.uni.wroc.pl .

Cite this article as:
A. G. Babenko, Yu. V. Kryakin , Modified Bernstein function and a uniform approximation of some rational fractions by polynomials, *Trudy Inst. Mat. Mekh. UrO RAN*, 2017, vol. 23, no. 3, pp. 43–57 .