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ON MULTIPLY MONOTONE FUNCTIONS

R. M. Trigub

The subject and the method of this paper belong to classical analysis. The Wiener Banach algebra (the normed ring) $A(\mathbb{R}^d)$, $d \in \mathbb{N}$, is the space of Fourier transforms of functions from $L_1(\mathbb{R}^d)$ (with pointwise product). The membership in this algebra is essential for Fourier multipliers from L_1 to L_1 and principal for the convergence on the space L_1 of summation methods for Fourier series and integrals given by one factor function. A function f is called m-multiply monotone on $\mathbb{R}_+ = (0, +\infty)$ if $(-1)^\nu f^{(\nu)}(t) \geq 0$ for $t \in \mathbb{R}_+$ and $0 \leq \nu \leq m+1$. For such functions, Shoenberg's integral presentation has long been known, which becomes Bernstein's formula for monotone functions as $m \to \infty$. Denote by $V_0(\mathbb{R}_+)$ the set of functions of bounded variation on $\mathbb{R}_+$, i.e., the set of functions representable as the difference of two bounded monotone functions. Denote by $V_m(\mathbb{R}_+)$, $m \in \mathbb{N}$, the space of functions f from $V_{0,\mathrm{loc}}(\mathbb{R}_+)$ such that $\|f\|_{V_m} = \sup_{t\in\mathbb{R}_+} |f(t)| + \int_0^\infty t^m |df^{(m)}(t)| < \infty$. This is a Banach algebra. A function f belongs to $V_m(\mathbb{R}_+)$ if and only if f can be represented as the difference of two bounded functions with convex derivatives of order $m-1$ (Theorem 1). We also study conditions under which functions of the form $f_0(|x|_{p,d})$, where $|x|_{p,d} = \big(\sum_{j=1}^d |x_j|^p\big)^{1/p}$, $x = (x_1, \ldots, x_d)$, for $p \in (0, \infty)$ and $|x|_\infty = \max\limits_{1\leq j\leq d} |x_j|$, belong to $A(\mathbb{R}^d)$. The case $p = 2$ (radial functions) is well studied, including the Pólya–Askey criterion of the positive definiteness of functions on $\mathbb{R}^d$. We prove Theorem 2, which has the following corollaries.

(1) If $f_0 \in C_0[0, \infty)$ and $f_0 \in V_d(\mathbb{R}_+)$, then $f_0(|x|_{p,d}) \in A(\mathbb{R}^d)$ for $p \in [1, \infty]$.

(2) If $f_0 \in C_0[0, \infty)$ and $f_0 \in V_{d+1}(\mathbb{R}_+)$, then $f_0(|x|_{p,d}) \in A(\mathbb{R}^d)$ for $p \in (0, 1)$.

We give some examples, including an example with an oscillating function.

Keywords: function of bounded variation, convex function, multiply monotone function, completely monotone function, positive definite function, Fourier transform.

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Roald Mikhailovich Trigub, Dr. Phis.-Math. Sci., Prof., Sumy State University, Sumy, 40007, Ukraine, e-mail: roald.trigub@gmail.com

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