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THE DIRECT THEOREM OF THE THEORY OF APPROXIMATION OF PERIODIC FUNCTIONS WITH MONOTONE FOURIER COEFFICIENTS IN DIFFERENT METRICS

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We study the problem of order optimality of an upper bound for the best approximation in $L_q(\mathbb{T})$ in terms of the lth-order modulus of smoothness (the modulus of continuity for $l=1$) in

$$L_p(\mathbb{T})\colon E_{n-1}(f)_q \le C(l,p,q)\Big(\sum_{\nu=n+1}^{\infty} \nu^{q\sigma-1}\omega_l^q(f;\pi/\nu)_p\Big)^{1/q}, \quad n\in\mathbb{N},$$

on the class $M_p(\mathbb{T})$ of all functions $f\in L_p(\mathbb{T})$ whose Fourier coefficients satisfy the conditions

$a_0(f)=0,\ a_n(f)\downarrow 0$, and $b_n(f)\downarrow 0\ (n\uparrow\infty)$, where $l\in\mathbb{N},\ 1<p<q<\infty,\ l>\sigma=1/p-1/q$, and $\mathbb{T}=(-\pi,\pi]$.

For $l=1$ and $p\ge 1$, the bound was first established by P. L. Ul'yanov in the proof of the inequality of different metrics for moduli of continuity; for $l>1$ and $p\ge 1$, the proof of the bound remains valid in view of the L_p-analog of the Jackson–Stechkin inequality. Below we formulate the main results of the paper. A function $f\in M_p(\mathbb{T})$ belongs to $L_q(\mathbb{T})$, where $1<p<q<\infty$, if and only if $\sum_{n=1}^{\infty} n^{q\sigma-1}\omega_l^q(f;\pi/n)_p<\infty$, and the following order inequalities hold:

(a) $E_{n-1}(f)_q + n^{\sigma}\omega_l(f;\pi/n)_p \asymp \big(\sum_{\nu=n+1}^{\infty}\nu^{q\sigma-1}\omega_l^q(f;\pi/\nu)_p\big)^{1/q}$, $n\in\mathbb{N}$;

(b) $n^{-(l-\sigma)}\big(\sum_{\nu=1}^{n}\nu^{p(l-\sigma)-1}E_{\nu-1}^p(f)_q\big)^{1/p} \asymp \big(\sum_{\nu=n+1}^{\infty}\nu^{q\sigma-1}\omega_l^q(f;\pi/\nu)_p\big)^{1/q}$, $n\in\mathbb{N}$.

In the lower bound in inequality (a), the second term $n^{\sigma}\omega_l(f;\pi/n)_p$ generally cannot be omitted. However, if the sequence $\{\omega_l(f;\pi/n)_p\}_{n=1}^{\infty}$ or the sequence $\{E_{n-1}(f)_p\}_{n=1}^{\infty}$ satisfies Bari's $(B_l^{(p)})$-condition, which is equivalent to Stechkin's (S_l)-condition, then

$$E_{n-1}(f)_q \asymp \Big(\sum_{\nu=n+1}^{\infty}\nu^{q\sigma-1}\omega_l^q(f;\pi/\nu)_p\Big)^{1/q}, \quad n\in\mathbb{N}.$$

The upper bound in inequality (b), which holds for any function $f\in L_p(\mathbb{T})$ if the series converges, is a strengthened version of the direct theorem. The order inequality (b) shows that the strengthened version is order-exact on the whole class $M_p(\mathbb{T})$.

Keywords: best approximation, modulus of smoothness, direct theorem in different metrics, trigonometric Fourier series with monotone coefficients, order-exact inequality on a class.

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