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INTEGRABILITY AND INVARIANT ALGEBRAIC CURVES FOR A CLASS OF KOLMOGOROV SYSTEMS

Rachid Boukoucha

There are many natural phenomena which can be modeled by Kolmogorov systems such as mathematical ecology, population dynamics, etc..

One of the more classical problems in the qualitative theory of planar differential systems is to characterize the existence or not of first integrals. For a two dimensional system the existence of a first integral completely determines its phase portrait. The question to determine the invariant algebraic curves of a given planar vector field, or to decide that no such curves exist, is part of a problem set forth by Poincaré. There are strong relationships between the integrability of **a** system, and its number of invariant algebraic curves. It is shown that the existence of a certain number of algebraic curves for a system implies its Darboux integrability, that is the first integral is the product of the algebraic solutions with complex exponents. The study of the number and location of limit cycles is one of the most important topics which is related to the second part of the unsolved Hilbert 16th problem. In this paper we introduce an explicit expression of invariant algebraic curves, then we prove that these systems are Darboux integrable and introduce an explicit expression of a Liouvillian first integral. Then we discuss the possibility of existence and non-existence of limit cycles of the two dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\left(\dfrac{P(x,y)}{Q(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \\ y' = y\left(\dfrac{R(x,y)}{S(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \end{cases}$$

where $P(x,y), Q(x,y), R(x,y), S(x,y), N(x,y)$ and $M(x,y)$ are homogeneous polynomials of degree n, m, n, m, a and a, respectively. The elementary method used in this paper seems to be fruitful to investigate more general planar differential Kolmogorov systems of ODEs in order to obtain explicit expression of invariant algebraic curves and for first integrals in order to characterize their trajectories. Finally, we discuss the possibility of existence and non-existence of limit cycles.

Keywords: Kolmogorov System, first integral, invariant algebraic curves, limit cycle, Hilbert 16th problem.

Рашид Букуша. Интегрируемость и инвариантные алгебраические кривые для класса систем Колмогорова.

Многие природные явления, например, явления, изучаемые в математической экологии и популяционной динамике, могут моделироваться системами Колмогорова. Одной из классических задач качественной теории двумерных дифференциальных систем является вопрос существования первого интеграла системы. Для двумерной системы ее фазовый портрет полностью определяется существованием первого интеграла. Вопрос существования и нахождения инвариантных алгебраических кривых заданного плоского векторного поля является частью набора задач, поставленных А. Пуанкаре. Между интегрируемостью системы и количеством ее инвариантных алгебраических кривых есть тесная связь. Доказано, что из существования у системы некоторого количества алгебраических кривых следует ее интегрируемость по Дарбу, т. е. первый интеграл является произведением алгебраических решений с комплексными показателями. Одним из важнейших вопросов, связанных со второй частью нерешенной 16-й проблемы Гильберта, является изучение количества и расположения предельных циклов. В данной работе дано явное выражение для инвариантных алгебраических кривых. Доказано, что эти системы являются интегрируемыми по Дарбу, и приведено явное выражение для первого интеграла Лиувилля. Затем обсуждается возможность существования предельных циклов двух систем Колмогорова следующего вида:

$$\begin{cases} x' = x\left(\dfrac{P(x,y)}{Q(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \\ y' = y\left(\dfrac{R(x,y)}{S(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \end{cases}$$

где $P(x,y)$, $Q(x,y)$, $R(x,y)$, $S(x,y)$, $N(x,y)$ и $M(x,y)$ — неоднородные многочлены степени n, m, n, m, a b a, соответственно. Представляется возможным использовать простой метод, предложенный в статье, для исследования более общих двумерных систем дифференциальных уравнений колмогоровского типаи получения явных выражений для инвариантных алгебраических кривых, а также первых интегралов, описывающих траектории системы. В работе также обсуждается возможность существования предельных циклов.

Ключевые слова: система Колмогорова, первый интеграл, инвариантные алгебраические кривые, предельный цикл, 16-я проблема Гильберта.

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1. Introduction

Many mathematical models in biology science and population dynamics, frequently involve the systems of ordinary differential equations having the form

$$\begin{cases} x' = \dfrac{dx}{dt} = x(t)\,F(x(t), y(t)), \\ y' = \dfrac{dy}{dt} = y(t)\,G(x(t), y(t)), \end{cases} \tag{1.1}$$

where F, G are two functions in the variables x and y, $x(t)$ and $y(t)$ represent the population density of two species at time t, and $F(x,y)$, $G(x,y)$ are the capita growth rate of each specie, usually, such systems are called Kolmogorov systems. Kolmogorov models are widely used in ecology to describe the interaction between two populations, and a limit cycle corresponds to an equilibrium state of the system. There are many natural phenomena which can be modeled by the Kolmogorov systems such as mathematical ecology and population dynamics [14; 18; 20; 21] chemical reactions, plasma physics [15], hydrodynamics [5], economics, etc..

In the classical Lotka–Volterra–Gause model, F and G are linear and it is well known that there are no limit cycles [11; 16]. There can, of course, only be one critical point in the interior of the realistic quadrant $(x > 0, y > 0)$ in this case, but this can be a center, however, there are no isolated periodic solutions. We remind that in the phase plane, a limit cycle of system (1.1) is an isolated periodic orbit in the set of all periodic orbits of system (1.1). There exist three main open problems in the qualitative theory of real planar differential systems [1; 2; 4; 10; 17; 19], the distinction between a centre and a focus, the determination of the number of limit cycles and their distribution, and the determination of its integrability. The determination of the number of limit cycles most important topics is related to the second part of the unsolved Hilbert 16th problem [13]. The importance for searching first integrals of a given system was already noted by Poincaré in his discussion on a method to obtain polynomial or rational first integrals [23].

System (1.1) is integrable on an open set Ω of $\mathbb{R}^2$ if there exists a non constant C^1 function $H : \Omega \to \mathbb{R}$, called a first integral of the system on Ω , which is constant on the trajectories of the system (1.1) contained in Ω, i.e. if

$$\frac{dH(x,y)}{dt} = \frac{\partial H(x,y)}{\partial x} xF(x,y) + \frac{\partial H(x,y)}{\partial y} yG(x,y) \equiv 0 \text{ in the points of } \Omega.$$

Moreover, $H = h$ is the general solution of this equation, where h is an arbitrary constant. One of the classical tools in the classification of all trajectories of a dynamical system is to find first integrals, for a two dimensional system the existence of a first integral completely determines its phase portrait. Of course, the easiest planar integrable systems are the Hamiltonian ones. The planar integrable systems which are not Hamiltonian can be in general very difficult to detect, for or more details about first integral see for instance [3; 12] . It is well known that for differential systems defined on the plane $\mathbb{R}^2$ the existence of a first integral determines their phase portrait [6].

A real or complex polynomial $U(x,y)$ is called algebraic solution of the polynomial differential system (1.1) if

$$\frac{\partial U(x,y)}{\partial x} xF(x,y) + \frac{\partial U(x,y)}{\partial y} yG(x,y) = K(x,y)\,U(x,y),$$

for some polynomial $K(x,y)$, called the cofactor of $U(x,y)$. The corresponding cofactor of $U(x,y)$ is always polynomial whether $U(x,y)$ is algebraic or non algebraic. If U is real, the curve $U(x,y)=0$ is an invariant under the flow of differential system (1.1) and the set $\{(x,y)\in\mathbb{R}^2,\ U(x,y)=0\}$ is formed by orbits of system (1.1). There are strong relationships between the integrability of system (1.1) and its number of invariant algebraic solutions. It is shown [9] that the existence of a certain number of algebraic solutions for a system implies the Darboux integrability of the system, that is the first integral is the product of the algebraic solutions with complex exponents [7; 8]. In [22], it is proved that, if a polynomial system (1.1) has a Liouvillian first integral, then it can be computed by using the invariant algebraic solutions and the exponential factors of the systems (1.1).

In this paper we introduce an explicit expression of invariant algebraic curves, then we proved that these systems are Darboux integrable and introduced an explicit expression of a Liouvillian first integral of the two dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\Big(\dfrac{P(x,y)}{Q(x,y)} + \ln\Big|\dfrac{N(x,y)}{M(x,y)}\Big|\Big), \\ y' = y\Big(\dfrac{R(x,y)}{S(x,y)} + \ln\Big|\dfrac{N(x,y)}{M(x,y)}\Big|\Big), \end{cases} \tag{1.2}$$

where $P(x,y), Q(x,y), R(x,y), S(x,y), N(x,y)$ and $M(x,y)$ are homogeneous polynomials of degree n, m, n, m, a and a, respectively.

We define the trigonometric functions

$$f_1(\theta) = \cos^2\theta\frac{P(\cos\theta,\sin\theta)}{Q(\cos\theta,\sin\theta)} + \sin^2\theta\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)},\quad f_2(\theta) = \ln\Big|\frac{N(\cos\theta,\sin\theta)}{M(\cos\theta,\sin\theta)}\Big|,$$

$$f_3(\theta) = \cos\theta\sin\theta\frac{R(\cos\theta,\sin\theta)}{S(\cos\theta,\sin\theta)} - \cos\theta\sin\theta\frac{P(\cos\theta,\sin\theta)}{Q(\cos\theta,\sin\theta)}.$$

2. Main result

Our main result on the expression of invariant algebraic curves and the existence of a Darboux first integral and the periodic orbits of the Kolmogorov system (1.2) is the following.

Theorem. *Consider a Kolmogorov system* (1.2), *then the following statements hold.*

(h_1) *If* $Q(x,y)S(x,y)\neq 0$ *and* $N(x,y)M(x,y)>0$ *then the curve*

$$U(x,y) = xy\frac{R(x,y)}{S(x,y)} - xy\frac{P(x,y)}{Q(x,y)} = 0$$

is an invariant algebraic curve of system (1.2).

(h_2) *If* $f_3(\theta)\neq 0$, $Q(\cos\theta,\sin\theta)S(\cos\theta,\sin\theta)\neq 0$, $N(\cos\theta,\sin\theta)M(\cos\theta,\sin\theta)>0$ *and* $n-m\neq 1$, *then the system* (1.2) *has the first integral*

$$H(x,y) = \left(x^2+y^2\right)^{\frac{n-m}{2}}\exp\left((m-n)\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\right) - (n-m)\int_0^{\arctan\frac{y}{x}}\exp\left((m-n)\int_0^{s} A(\omega)\,d\omega\right)B(s)\,ds, \tag{2.1}$$

where $A(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$ *and* $B(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$.

Moreover, the system (1.2) *has no limit cycle.*

(h_3) *If* $f_3(\theta) \neq 0$, $Q(\cos\theta,\sin\theta) S(\cos\theta,\sin\theta) \neq 0$, $N(\cos\theta,\sin\theta) M(\cos\theta,\sin\theta) > 0$ *and* $n-m=1$, *then the system* (1.2) *has the first integral*

$$H(x,y) = \sqrt{(x^2+y^2)}\exp\Big(-\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\Big) - \int_0^{\arctan\frac{y}{x}} \exp\Big(-\int_0^{s} A(\omega)\,d\omega\Big) B(s)\,ds. \qquad (2.2)$$

Moreover, the system (1.2) *has no limit cycle.*

(h_4) *If* $f_3(\theta) = 0$ *for all* $\theta \in \mathbb{R}$, *then the system* (1.2) *has the first integral*

$$H(x,y) = \frac{y}{x}.$$

Moreover, the system (1.2) *has no limit cycle.*

Proof.

Proof of statement (h_1). *Suppose that* $Q(x,y) S(x,y) \neq 0$ and $N(x,y) M(x,y) > 0$.

We prove that $U(x,y) = xy\dfrac{R(x,y)}{S(x,y)} - xy\dfrac{P(x,y)}{Q(x,y)} = 0$ is an invariant algebraic curve of the differential system (1.2).

Indeed, we have

$$\frac{\partial U}{\partial x}x\Big(\ln\Big|\frac{N}{M}\Big| + \frac{P}{Q}\Big) + \frac{\partial U}{\partial y}y\Big(\ln\Big|\frac{N}{M}\Big| + \frac{R}{S}\Big) = \frac{\partial U}{\partial x}x\ln\Big|\frac{N}{M}\Big| + \frac{\partial U}{\partial y}y\ln\Big|\frac{N}{M}\Big| + \frac{\partial U}{\partial x}x\frac{P}{Q} + \frac{\partial U}{\partial y}y\frac{R}{S}.$$

Then, taking into account that if $P(x,y), Q(x,y), R(x,y)$ and $S(x,y)$ are homogeneous polynomials of degree n, m, n, m respectively, we have

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = nP, \quad x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} = mQ, \quad x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = nR \quad \text{and} \quad x\frac{\partial S}{\partial x} + y\frac{\partial S}{\partial y} = mS.$$

Then, we have

$$\frac{\partial U}{\partial x}x\ln\Big|\frac{N}{M}\Big| + \frac{\partial U}{\partial y}y\ln\Big|\frac{N}{M}\Big| = x\frac{\partial}{\partial x}\Big(xy\frac{R}{S} - xy\frac{P}{Q}\Big)\ln\Big|\frac{N}{M}\Big| + y\frac{\partial}{\partial y}\Big(xy\frac{R}{S} - xy\frac{P}{Q}\Big)\ln\Big|\frac{N}{M}\Big|$$

$$= xy\Big(2\frac{R}{S} - 2\frac{P}{Q} + \frac{P(xQ_x + yQ_y) - Q(xP_x + yP_y)}{Q^2} + \frac{S(xR_x + yR_y) - R(xS_x + yS_y)}{S^2}\Big)\ln\Big|\frac{N}{M}\Big|$$

$$= xy\Big(2\frac{R}{S} - 2\frac{P}{Q} + \frac{mP - nP}{Q} + \frac{nR - mR}{S}\Big)\ln\Big|\frac{N}{M}\Big|$$

$$= (n-m+2)\,xy\Big(\frac{R}{S} - \frac{P}{Q}\Big)\ln\Big|\frac{N}{M}\Big| = (n-m+2)\,U\ln\Big|\frac{N}{M}\Big|.$$

On the other hand, substituting

$$y\frac{\partial P}{\partial y} = nP - x\frac{\partial P}{\partial x}, \quad y\frac{\partial Q}{\partial y} = mQ - x\frac{\partial Q}{\partial x}, \quad x\frac{\partial R}{\partial x} = nR - y\frac{\partial R}{\partial y} \quad \text{and} \quad x\frac{\partial S}{\partial x} = mS - y\frac{\partial S}{\partial y},$$

in what follows, we get

$$\frac{\partial U}{\partial x}x\frac{P}{Q} + \frac{\partial U}{\partial y}y\frac{R}{S} = \frac{\partial\Big(xy\frac{R}{S} - xy\frac{P}{Q}\Big)}{\partial x}x\frac{P}{Q} + \frac{\partial\Big(xy\frac{R}{S} - xy\frac{P}{Q}\Big)}{\partial y}y\frac{R}{S}$$

$$= xy\left(\frac{R^2}{S^2} - \frac{P^2}{Q^2} + x\frac{SR_x - RS_x}{S^2}\frac{P}{Q} - x\frac{QP_x - PQ_x}{Q^2}\frac{P}{Q} + y\frac{SR_y - RS_y}{S^2}\frac{R}{S} - y\frac{QP_y - PQ_y}{Q^2}\frac{R}{S}\right)$$

$$= xy\left(\frac{R^2}{S^2} - \frac{P^2}{Q^2} + \frac{xS^2PR_x - xSRPS_x + yQSRR_y - yQR^2S_y}{S^3Q}\right.$$
$$\left. + \frac{-xSQPP_x + xSP^2Q_x - yRQ^2P_y + yQRPQ_y}{Q^3S}\right)$$

$$= xy\left(\frac{R^2}{S^2} - \frac{P^2}{Q^2} + \frac{-xS^3P\left(QP_x - PQ_x\right) + xRQS^2\left(QP_x - PQ_x\right) - nS^2RQ^2P + mS^2Q^2RP}{S^3Q^3}\right.$$
$$\left. + \frac{-ySPQ^2\left(SR_y - RS_y\right) + yQ^3R\left(SR_y - RS_y\right) + nS^2Q^2PR - mS^2RPQ^2}{S^3Q^3}\right)$$
$$= xy\Big[\frac{R^2}{S^2} - \frac{P^2}{Q^2} - x\frac{P}{Q}\frac{\left(QP_x - PQ_x\right)}{Q^2} + x\frac{R}{S}\frac{\left(QP_x - PQ_x\right)}{Q^2} - y\frac{P}{Q}\frac{SR_y - RS_y}{S^2} + y\frac{R}{S}\frac{SR_y - RS_y}{S^2}\Big]$$
$$= xy\Big[\frac{R^2}{S^2} - \frac{P^2}{Q^2} - \Big(x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big)\frac{P}{Q} + \Big(x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big)\frac{R}{S}\Big]$$
$$= xy\Big[\frac{R^2}{S^2} - \frac{P^2}{Q^2} + \Big(x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big)\Big(\frac{R}{S} - \frac{P}{Q}\Big)\Big]$$
$$= xy\Big(\frac{R}{S} - \frac{P}{Q}\Big)\Big[\frac{R}{S} + \frac{P}{Q} + x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big] = U\Big[\frac{R}{S} + \frac{P}{Q} + x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big].$$

In short, we have

$$\frac{\partial U}{\partial x}x\Big(\ln\Big|\frac{N}{M}\Big| + \frac{P}{Q}\Big) + \frac{\partial U}{\partial y}y\Big(\ln\Big|\frac{N}{M}\Big| + \frac{R}{S}\Big)$$
$$= \Big[\Big(n - m + 2\Big)\ln\Big|\frac{N}{M}\Big| + \frac{P}{Q} + \frac{R}{S} + x\frac{\partial}{\partial x}\Big(\frac{P}{Q}\Big) + y\frac{\partial}{\partial y}\Big(\frac{R}{S}\Big)\Big]U.$$

Therefore, $U\left(x,y\right) = xy\dfrac{R\left(x,y\right)}{S\left(x,y\right)} - xy\dfrac{P\left(x,y\right)}{Q\left(x,y\right)} = 0$ is an invariant algebraic curve of the polynomial differential systems (1.2) with the the cofactor

$$K\left(x,y\right) = \left(n - m + 2\right)\ln\Big|\frac{N\left(x,y\right)}{M\left(x,y\right)}\Big| + \frac{P\left(x,y\right)}{Q\left(x,y\right)} + \frac{R\left(x,y\right)}{S\left(x,y\right)} + x\frac{\partial}{\partial x}\left(\frac{P\left(x,y\right)}{Q\left(x,y\right)}\right) + y\frac{\partial}{\partial y}\Big(\frac{R\left(x,y\right)}{S\left(x,y\right)}\Big).$$

Hence, statement (h_1) is proved. □

Proof of statement (h_2), (h_3) and (h_4). In order to prove our results (h_2), (h_3) and (h_4) we write the polynomial differential system (1.2) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system (1.2) becomes

$$\begin{cases} r' = f_1\left(\theta\right)r^{n-m+1} + f_2\left(\theta\right)r, \\ \theta' = f_3\left(\theta\right)r^{n-m}, \end{cases} \tag{2.3}$$

where the trigonometric functions $f_1\left(\theta\right)$, $f_2\left(\theta\right)$, $f_3\left(\theta\right)$ are given in introduction,

$$r' = \frac{dr}{dt} \quad \text{and} \quad \theta' = \frac{d\theta}{dt}.$$

Suppose that

$$f_3\left(\theta\right) \neq 0, \quad Q\left(\cos\theta, \sin\theta\right)S\left(\cos\theta, \sin\theta\right) \neq 0, \quad N\left(\cos\theta, \quad \sin\theta\right)M\left(\cos\theta, \sin\theta\right) > 0 \;\text{ and }\; n - m \neq 1.$$

Taking as independent variable the coordinate θ, this differential system (2.3) writes

$$\frac{dr}{d\theta} = A(\theta)\, r + B(\theta)\, r^{1-n+m}, \tag{2.4}$$

where $A(\theta) = \dfrac{f_1(\theta)}{f_3(\theta)}$ and $B(\theta) = \dfrac{f_2(\theta)}{f_3(\theta)}$, which is a Bernoulli equation. By introducing the standard change of variables $\rho = r^{n-m}$ we obtain the linear equation

$$\frac{d\rho}{d\theta} = (n-m)\left(A(\theta)\,\rho + B(\theta)\right). \tag{2.5}$$

The general solution of linear equation (2.5) is

$$\rho(\theta) = \exp\left((n-m)\int_0^{\theta} A(\omega)\, d\omega\right)\left[\mu + (n-m)\int_0^{\theta} \exp\left((m-n)\int_0^{s} A(\omega)\, d\omega\right) B(s)\, ds\right],$$

where $\mu \in \mathbb{R}$, which has the first integral (2.1).

Since this first integral is a function that can be expressed by quadratures of elementary functions, it is a Liouvillian function, and consequently system (1.2) is Darboux integrable.

Let Γ be a periodic orbit surrounding an equilibrium located in one of the open quadrants, and let $h_\Gamma = H(\Gamma)$.

The curves $H = h$ with $h \in \mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates are written as

$$x^2 + y^2 = \left[h \exp\left((n-m)\int_0^{\arctan \frac{y}{x}} A(\omega)\, d\omega\right)\right.$$
$$\left. + (n-m)\exp\left((n-m)\int_0^{\arctan \frac{y}{x}} A(\omega)\, d\omega\right)\int_0^{\arctan \frac{y}{x}} \exp\left((m-n)\int_0^{s} A(\omega)\, d\omega\right) B(s)\, ds\right]^{\frac{2}{n-m}},$$

where $h \in \mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$x^2 + y^2 = \left[h_\Gamma \exp\left((n-m)\int_0^{\arctan \frac{y}{x}} A(\omega)\, d\omega\right)\right.$$
$$\left. + (n-m)\exp\left((n-m)\int_0^{\arctan \frac{y}{x}} A(\omega)\, d\omega\right)\int_0^{\arctan \frac{y}{x}} \exp\left((m-n)\int_0^{s} A(\omega)\, d\omega\right) B(s)\, ds\right]^{\frac{2}{n-m}}.$$

But this curve cannot contain the periodic orbit Γ and consequently no limit cycle contained in the realistic quadrant $(x > 0, y > 0)$, because this curve in realistic quadrant has at most a unique point on every straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y = \eta x$ for all $\eta \in \left]0, +\infty\right[$, the abscissa is given by

$$x = \frac{1}{\sqrt{1+\eta^2}}\left[h \exp\left((n-m)\int_0^{\arctan \eta} A(\omega)\, d\omega\right)\right.$$
$$\left. + (n-m)\exp\left((n-m)\int_0^{\arctan \eta} A(\omega)\, d\omega\right)\int_0^{\arctan \eta} \exp\left((m-n)\int_0^{s} A(\omega)\, d\omega\right) B(s)\, ds\right]^{\frac{1}{n-m}},$$

at most a unique value of x on every half straight OX^+, consequently at most a unique point in realistic quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit, consequently no limit cycle.

Hence statement (h_2) is proved. □

Suppose now that

$$f_3(\theta)\neq 0,\quad Q(\cos\theta,\sin\theta)\,S(\cos\theta,\sin\theta)\neq 0,\quad N(\cos\theta,\quad \sin\theta)\,M(\cos\theta,\sin\theta)>0\ \text{ and }\ n-m=1.$$

Taking as independent variable the coordinate θ, this differential system (2.3) writes

$$\frac{dr}{d\theta}=A(\theta)\,r+B(\theta)\,. \tag{2.6}$$

The general solution of linear equation (2.6) is

$$r(\theta)=\exp\Big(\int_0^\theta A(\omega)\,d\omega\Big)\Big[\mu+\int_0^\theta \exp\Big(-\int_0^s A(\omega)\,d\omega\Big)B(s)\,ds\Big],$$

where $\mu\in\mathbb{R}$, which has the first integral (2.2).

Let Γ be a periodic orbit surrounding an equilibrium located in one of the realistic quadrant $(x>0, y>0)$, and let $h_\Gamma = H(\Gamma)$.

The curves $H=h$ with $h\in\mathbb{R}$, which are formed by trajectories of the differential system (1.2), in Cartesian coordinates are written as

$$\sqrt{x^2+y^2}=h\exp\Big(\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\Big)+\exp\Big(\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\Big)\int_0^{\arctan\frac{y}{x}}\exp\Big(-\int_0^s A(\omega)\,d\omega\Big)B(s)\,ds,$$

where $h\in\mathbb{R}$.

Therefore the periodic orbit Γ is contained in the curve

$$\sqrt{x^2+y^2}=h_\Gamma\exp\Big(\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\Big)+\exp\Big(\int_0^{\arctan\frac{y}{x}} A(\omega)\,d\omega\Big)\int_0^{\arctan\frac{y}{x}}\exp\Big(-\int_0^s A(\omega)\,d\omega\Big)B(s)\,ds,$$

But this curve cannot contain the periodic orbit Γ, and consequently no limit cycle contained in the realistic quadrant $(x>0, y>0)$, because this curve in realistic quadrant has at most a unique point on every straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$.

To be convinced by this fact, one has to compute the abscissa points of intersection of this curve with straight line $y=\eta x$ for all $\eta\in\,]0,+\infty[$, the abscissa is given by

$$x=\frac{1}{\sqrt{x^2+y^2}}\Big[h_\Gamma\exp\Big(\int_0^{\arctan\eta} A(\omega)\,d\omega\Big)+\exp\Big(\int_0^{\arctan\eta} A(\omega)\,d\omega\Big)\int_0^{\arctan\eta}\exp\Big(-\int_0^s A(\omega)\,d\omega\Big)B(s)\,ds\Big],$$

at most a unique value of x on every half straight OX^+, consequently at most a unique point in realistic quadrant $(x>0, y>0)$. So this curve cannot contain the periodic orbit, consequently no limit cycle.

Hence statement (h_3) is proved. □

Assume now that $f_3(\theta)=0$ for all $\theta\in\mathbb{R}$, then from system (2.3) it follows that $\theta'=0$. So the straight lines through the origin of coordinates of the differential system (1.2) are invariant by the flow of this system. Hence, $\frac{y}{x}$ is a first integral of the system, then curves which are formed by the trajectories of the differential system (1.2), in Cartesian coordinates are written as $y-hx=0$, where $h\in\mathbb{R}$, since all straight lines through the origin are formed by trajectories, clearly the system has no periodic orbits, consequently no limit cycle.

This completes the proof of statement (h_4) and Theorem 1. □

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Rachid Boukoucha, Department of Technology, Faculty of Technology, University of Bejaia, 06000 Bejaia, Algeria, e-mail: rachid_boukecha@yahoo.fr .