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INTEGRABILITY AND INVARIANT ALGEBRAIC CURVES FOR A CLASS OF KOLMOGOROV SYSTEMS

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There are many natural phenomena which can be modeled by Kolmogorov systems such as mathematical ecology, population dynamics, etc..

One of the more classical problems in the qualitative theory of planar differential systems is to characterize the existence or not of first integrals. For a two dimensional system the existence of a first integral completely determines its phase portrait. The question to determine the invariant algebraic curves of a given planar vector field, or to decide that no such curves exist, is part of a problem set forth by Poincaré. There are strong relationships between the integrability of **a** system, and its number of invariant algebraic curves. It is shown that the existence of a certain number of algebraic curves for a system implies its Darboux integrability, that is the first integral is the product of the algebraic solutions with complex exponents. The study of the number and location of limit cycles is one of the most important topics which is related to the second part of the unsolved Hilbert 16th problem. In this paper we introduce an explicit expression of invariant algebraic curves, then we prove that these systems are Darboux integrable and introduce an explicit expression of a Liouvillian first integral. Then we discuss the possibility of existence and non-existence of limit cycles of the two dimensional Kolmogorov systems of the form

$$\begin{cases} x' = x\left(\dfrac{P(x,y)}{Q(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \\ y' = y\left(\dfrac{R(x,y)}{S(x,y)} + \ln\left|\dfrac{N(x,y)}{M(x,y)}\right|\right), \end{cases}$$

where $P(x,y), Q(x,y), R(x,y), S(x,y), N(x,y)$ and $M(x,y)$ are homogeneous polynomials of degree n, m, n, m, a and a, respectively. The elementary method used in this paper seems to be fruitful to investigate more general planar differential Kolmogorov systems of ODEs in order to obtain explicit expression of invariant algebraic curves and for first integrals in order to characterize their trajectories. Finally, we discuss the possibility of existence and non-existence of limit cycles.

Keywords: Kolmogorov System, First Integral, Invariant Algebraic Curves, Limit Cycle, Hilbert 16th Problem.

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