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STABILIZATION OF DISCRETE TIME SYSTEMS BY REFLECTION COEFFICIENTS

T. Büyükköroğlu, G. Çelebi, V. Dzhafarov

For single-input single-output discrete-time systems, we consider a stabilization problem by a fixed order controller. A number of examples show that such controller may not exist. It is assumed that the controller depends linearly on a stabilizing parameter. In this case, the stabilizing controller defines an affine subset in the parameter space. We use the well-known property of the Schur stability region in the parameter space. According to this property the closed convex hull of this region is a polytope with known vertices. Every stable vector has a preimage in the open cube $(-1,1)^n$, and this preimage is called the reflection coefficient of this stable polynomial. By using reflection coefficients and polytopic properties of the stability region we obtain the stabilizability condition. This condition is expressed in terms of vertices of the stability region which is a multilinear image of the cube of reflection coefficients.

Keywords: discrete system, stability, affine stabilizer, reflection coefficient.

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Рассматривается задача стабилизации дискретных систем с одним входом и одним выходом регулятором заданного порядка. Ряд примеров показывает, что такой регулятор может не существовать. Предполагается, что регулятор линейно зависит от стабилизирующих параметров. В этом случае стабилизирующий регулятор определяет аффинное подмножество в пространстве параметров. В этом пространстве замкнутая выпуклая оболочка области устойчивости по Шуру является многогранником с известными вершинами. Каждый стабильный вектор имеет прообраз в открытом кубе $(-1,1)^n$, и этот прообраз называется рефлективным коэффициентом соответствующего стабилизирующего полинома. На основе рефлективных коэффициентов и свойств многогранной области устойчивости получено условие стабилизируемости. Это условие выражено в терминах вершин области устойчивости, которая является мультилинейным образом куба рефлективных коэффициентов.

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1. Introduction

Consider n-th degree polynomial $p(s) = a_1 + a_2 s + \cdots + a_n s^{n-1} + a_{n+1} s^n$ with $a_{n+1} \neq 0$. This polynomial is called Hurwitz stable when all its roots lie in the open left half plane and Schur stable when all its roots lie in the open unit disc. Division by a_{n+1} does not affect the stability property, therefore, we will assume that $a_{n+1} = 1$, that is

$$p(s) = a_1 + a_2 s + \cdots + a_n s^{n-1} + s^n. \tag{1.1}$$

The polynomial (1.1) can be expressed as n-dimensional vector $p = (a_1, a_2, ..., a_n)^T \in \mathbb{R}^n$. Define the following subsets of $\mathbb{R}^n$:

$$\mathcal{H}_n = \{p \in \mathbb{R}^n : \text{ The polynomial (1.1) is Hurwitz stable}\},$$

$$\mathcal{S}_n = \{p \in \mathbb{R}^n : \text{The polynomial (1.1) is Schur stable}\}.$$

The set $\mathcal{H}_n$ $(n \geq 3)$ is open, nonconvex, unbounded, and the set $\mathcal{S}_n$ $(n \geq 3)$ is open, nonconvex and bounded [1–3]. In the case of $n = 2$, the set $\mathcal{H}_2$ equals $\{(a_1, a_2)\colon a_1 > 0,\ a_2 > 0\}$, and $\mathcal{S}_2$ is the open triangle with vertices $(-1,0)$, $(1,2)$, $(1,-2)$. In [3], it is shown that the closed convex hull of $\mathcal{S}_n$ is a polytope in $\mathbb{R}^n$ with known vertices, i.e.

$$\overline{\text{co}}\,\mathcal{S}_n = \text{co}\{v^1, v^2, \dots, v^{n+1}\}, \tag{1.2}$$

where v^i corresponds to the unstable polynomial $(s+1)^{i-1}(s-1)^{n-i+1}$ $(1 \le i \le n+1)$. In other words,

$$v^1(s) = (s-1)^n, \quad v^2(s) = (s-1)^{n-1}(s+1), \dots, v^{n+1}(s) = (s+1)^n.$$

For example, in the case of $n = 3$

$$v^1 = (-1,3,-3)^T, \quad v^2 = (1,-1,-1)^T, \quad v^3 = (-1,-1,1)^T, \quad v^4 = (1,3,3)^T.$$

Construction of $\mathcal{S}_n$ recursively starts from $\mathcal{S}_1$ and $\mathcal{S}_2$. It is given in [3].

Consider the transfer function

$$G(s) = \frac{n(s)}{d(s)}$$

and the stabilizer $C(s) = \dfrac{a(s,c)}{b(s,c)}$, where $c = (c_1, c_2, \dots, c_l)^T \in \mathbb{R}^l$ is a stabilizing parameter and $n(s)$, $d(s)$, $a(s,c)$, $b(s,c)$ are polynomials. It is assumed that $l < n$ and $a(s,c)$, $b(s,c)$ depend on vector c in the affine linear way.

The closed loop characteristic polynomial is

$$p(s,c) = n(s)a(s,c) + d(s)b(s,c) = p^0(s) + c_1 p^1(s) + \cdots + c_l p^l(s). \tag{1.3}$$

Additionally, we assume that degree$(p^0(s)) = n$, degree$(p^i(s)) < n$ $(i = 1,2,\dots,l)$. From these conditions it follows that $p(s,c)$ is an unitary polynomial. The vector $c \in \mathbb{R}^l$ is called stabilizing if the corresponding $p(s,c)$ is Schur stable. In this paper, we consider the case where the number of stabilizing parameters l equals $n-1$, where n is the degree of the characteristic polynomial.

Many works have been devoted to the problems of stabilization of continuous and discrete time systems (see [4–9] and references therein).

In [4], a large number of Schur stable polynomials are generated using the known methods. These polynomials are projected on the set of characteristic polynomials and, as a result, stabilizing controller parameters are determined. The same idea is developed in [5] where random generations of stable segments of polynomials are used for determination of the stabilizing parameter.

In [6], stabilization algorithms are given for continuous time systems, both deterministic and stochastic. In [7], stabilization algorithms based on linear programming are given for discrete time systems.

In [8], stabilization conditions are obtained by estimating the distance between the affine controller set and the Schur stability region $\mathcal{S}_n$.

Remark 1. *The characteristic polynomial of the type (1.3) with $l = n-1$ appears in the stabilization problem for linear time-invariant discrete system*

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t)$$

(A, B and C are real matrices of suitable dimension) with output feedback of the form $u(t) = Ky(t)$ and rank$(K) = 1$ *(see* [3, Introduction]*).*

2. Main result

In this section, we give the definition of reflection coefficients for Schur stability and necessary and sufficient conditions for stabilization in the case of $l = n-1$.

Reflection coefficients or Schur-Szegö parameters [10; 11] for polynomials have been widely used in the stability problems of discrete systems. For $k_i \in \mathbb{R}$ $(i = 1, 2, \dots, n)$ and $n \geq 3$ reflection map $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$(f_1, f_2, \dots, f_n)^T (k_1, \dots, k_n) = R_n(k_n) \begin{bmatrix} 0^T \\ R_{n-1}(k_{n-1}) \end{bmatrix} \cdots \begin{bmatrix} 0^T \\ R_1(k_1) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $R_j(k_j) = I_{j+1} + k_j E_{j+1}$, I_j is the $j \times j$ identity matrix, $j \times j$ matrix E_j is the following one:

$$E_j = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}.$$

The map f is multilinear ([11]), that is affine linear with respect to each component k_i. The explicit formulas for f are given in the special cases of $n = 3$ and $n = 4$:

$$f_1(k_1, k_2, k_3) = -k_3, \quad f_2(k_1, k_2, k_3) = -k_1k_2k_3 + k_1k_3 - k_2, \quad f_3(k_1, k_2, k_3) = k_1k_2 + k_2k_3 - k_1,$$

$$\begin{aligned} f_1(k_1, k_2, k_3, k_4) &= -k_4, \\ f_2(k_1, k_2, k_3, k_4) &= -k_1k_2k_4 - k_2k_3k_4 + k_1k_4 - k_3, \\ f_3(k_1, k_2, k_3, k_4) &= k_1k_2k_3k_4 - k_1k_2k_3 - k_1k_3k_4 + k_1k_3 + k_2k_4 - k_2, \\ f_4(k_1, k_2, k_3, k_4) &= k_1k_2 + k_2k_3 + k_3k_4 - k_1. \end{aligned}$$

According to [11], for arbitrary polynomial $f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$ there exist $k_1, k_2, \dots, k_n$ such that $f_1 = f_1(k_1, \dots, k_n), \dots, f_n = f_n(k_1, \dots, k_n)$.

The numbers $k_1, k_2, \dots, k_n$ are called the reflection coefficients of the polynomial $f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$. The following fact is important:

Proposition 1 [11]. *The unitary polynomial $p(s) = f_1 + f_2 s + \cdots + f_n s^{n-1} + s^n$ is Schur stable if and only if its reflection coefficients satisfy the conditions $|k_i| < 1$ $(i = 1, 2, \dots, n)$.*

According to the fact mentioned above, there exists a multilinear one to one map f from the open cube $(-1, 1)^n$ onto $\mathcal{S}_n$.

Define vectors $V^i \in \mathbb{R}^n$ $(i = 0, 1, \dots, n)$, where V^0 corresponds to $p^0(s)$ and V^i to $p^i(s)$ $(i = 1, 2, \dots, n)$. We add zero components for $p^i(s)$ $(i \geq 1)$ in order to complete n-dimension (see [8]).

For example, assume that $n = 4$, $l = 3$ and $p^0(s) = 1 + 2s - s^2 + s^3 + s^4$, $p^1(s) = 1 - 2s + s^2$, $p^2(s) = 1 + 2s$, $p^3(s) = 2 - s^2 + s^3$. Then $V^0 = (1, 2, -1, 1)^T$, $V^1 = (1, -2, 1, 0)^T$, $V^2 = (1, 2, 0, 0)^T$, $V^3 = (2, 0, -1, 1)^T$. Consider $n \times l$ matrix

$$A = \left[V^1, V^2, \dots, V^l \right].$$

From now, we assume that $V^1, V^2, \dots, V^l$ are linearly independent and $l = n - 1$. In this case, the family (1.3) corresponds TO $(n-1)$-dimensional affine subset $\mathcal{A} = \{Ac + V^0 \colon c \in \mathbb{R}^{n-1}\} \subset \mathbb{R}^n$, and there exists stabilizing vector c if and only if

$$\mathcal{A} \cap \mathcal{S}_n \neq \varnothing. \tag{2.4}$$

Since $\operatorname{rank}(A) = n - 1$ and $V^0 \neq 0$, the subset $\mathcal{A}$ is $(n-1)$-dimensional hyperplane which does not pass through the origin. Normal vector of $\mathcal{A}$ satisfies the following homogenous system

$$\langle N, V^1 \rangle = 0, \quad \langle N, V^2 \rangle = 0, \dots, \langle N, V^{n-1} \rangle = 0,$$

where the symbol $\langle \cdot, \cdot \rangle$ stands for the scalar product. The hyperplane has the equation

$$\langle N, x - V^0 \rangle = 0 \text{ or } \langle N, x \rangle = \alpha,$$

where $\alpha = \langle N, V^0 \rangle$.

Theorem 1. *Assume that $l = n-1$, and the vectors $V^1, V^2, \dots, V^{n-1}$ are linearly independent. There exists a stabilizing vector if and only if there exist vertices v^i, v^j of the polytope $\overline{\mathrm{co}}\,\mathcal{S}_n = \mathrm{co}\{v^1, v^2, \dots, v^{n+1}\}$ such that v^i and v^j lie in the opposite sides of the hyperplane $\mathcal{A}$. In other words, there exist v^i, v^j such that $\langle N, v^i\rangle > \alpha$, $\langle N, v^j\rangle < \alpha$.*

Proof.$\Leftarrow$). By the known property of a multilinear function defined on a box ([2, p. 247]), there exist vertices k^i and k^j of the cube $[-1,1]^n$ such that $f(k^i) = v^i$, $f(k^j) = v^j$. Consider a curve $\mathcal{L}$ connecting k^i and k^j which is contained in $(-1,1)^n$ with the exception of the points k^i and k^j. The image $f(\mathcal{L})$ intersects the hyperplane $\mathcal{A}$, since their end points v^i and v^j lie in the opposite sides of $\mathcal{A}$. Indeed, assume that $f(\mathcal{L})$ has equation $x = x(t)$ $(0 \le t \le 1)$. Consider scalar function $b(t) = \langle N, x(t)\rangle$. Then $b(0) = \langle N, x(0)\rangle = \langle N, v^i\rangle > \alpha$, $b(1) = \langle N, x(1)\rangle = \langle N, v^j\rangle < \alpha$ and by continuity there exists $t_* \in (0,1)$ such that $b(t_*) = \alpha$, i.e. $\langle N, x(t_*)\rangle = \alpha$ or $x(t_*) \in \mathcal{A}$.
$\Rightarrow$). Let c be a stabilizing parameter. Then the hyperplane $\langle N, x\rangle = \alpha$ intersects the set $\mathcal{S}_n$: $\mathcal{A} \cap \mathcal{S}_n \neq \emptyset$. By the contrary, assume that $\langle N, v^i\rangle \ge \alpha$ for all $i = 1, 2, \dots, n+1$. Then the closed convex hull $\overline{\mathrm{co}}\,\mathcal{S}_n = \mathrm{co}\{v^1, v^2, \dots, v^{n+1}\}$ is contained in the half space $\{x : \ \langle N, x\rangle \ge \alpha\}$. From this and the openness property of $\mathcal{S}_n$, it follows that

$$\mathcal{A} \cap \mathcal{S}_n = \varnothing$$

which is a contradiction. This contradiction proves the necessity.

Since the hyperplane $\mathcal{A}$ does not pass through the origin then the following corollary is true

Corollary 1. *Let all conditions of Theorem 1 be satisfied. Then there exists a stabilizing vector c if and only if there exists vertex v^i such that v^i and the origin lie in opposite sides of $\mathcal{A}$.*

3. Evaluation of stabilizing parameter

Theorem 1 indicates a way for evaluation of a stabilizing parameter. Assume that all conditions of Theorem 1 are satisfied. As noted above, the hyperplane $\mathcal{A}$ does not contain the origin, therefore, there exists vertex v^i such that v^i and the origin lie in the opposite sides of $\mathcal{A}$ (Corollary 1). Consider line segment $\mathcal{C}$ which connects the vertex k^i and the origin, where k^i is the preimage of v^i, that is $f(k^i) = v^i$. The segment $\mathcal{C}$ is defined by

$$k_j(t) = \begin{cases} t & \text{if } k^i_j = 1 \\ -t & \text{if } k^i_j = -1 \end{cases} \qquad (j = 1, 2, \dots, n).$$

The image $f(\mathcal{C})$ is contained in $\mathcal{S}_n$, except the point $f(v^i)$. The curve$f(\mathcal{C}) \subset \mathbb{R}^n$ depends on the parameter $t \in [0,1]$ and has the equation $x = \varphi(t)$. After inserting $x = \varphi(t)$ into equation of $\mathcal{A}$, we have the scalar equation with respect to t

$$\langle N, \varphi(t)\rangle = \alpha, \tag{3.5}$$

from which the values $t_* \in (0,1)$ and $x_* = \varphi(t_*)$ can be calculated. Finally, the value of c can be defined from the following system of linear equations

$$Ac + V^0 = x_*. \tag{3.6}$$

Example 1. Consider the transfer function and the stabilizer

$$G(s) = \frac{s-1}{s^3 + 2s^2 + s}, \quad C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s}.$$

The closed loop system has the following characteristic polynomial

$$p(s,c) = s^4 + 2s^3 + s^2 + c_1(s^3 - s^2) + c_2(s^2 - s) + c_3(s - 1).$$

Here $V^0 = (0,0,1,2)^T$, $V^1 = (0,0,-1,1)^T$, $V^2 = (0,-1,1,0)^T$, $V^3 = (-1,1,0,0)^T$. The vectors V^1, V^2, V^3 are linearly independent and $N = (1,1,1,1)^T$ is a normal vector. The hyperplane $\mathcal{A}$ has equation $x_1 + x_2 + x_3 + x_4 - 3 = 0$. The stability set $\mathcal{S}_4$ has five vertices: $v^1 = (1,4,6,4)^T$, $v^2 = (-1,-2,0,2)^T$, $v^3 = (1,0,-2,0)^T$, $v^4 = (-1,2,0,-2)^T$, $v^5 = (1,-4,6,-4)^T$. The vertex v^1 and the origin lie in the opposite sides of $\mathcal{A}$. Vertex v^1 is the image of the vertex $k^1 = (-1,-1,-1,-1)^T$ of the cube $[-1,1]^4$. The line segment $\mathcal{C}$ connecting k^1 and $(0,0,0,0)^T$ has equation $k_j(t) = -t$ $(j = 1,2,3,4)$. The image of $\mathcal{C}$ under f is the following curve in $\mathbb{R}^4$:

$$x_1(t) = t, \quad x_2(t) = 2t^3 + t^2 + t, \quad x_3(t) = t^4 + 2t^3 + t^2 + t, \quad x_4(t) = 3t^2 + t \quad (0 \le t \le 1).$$

For the point of intersection of $f(\mathcal{C})$ and $\mathcal{A}$ we have the following equation $(t+1)^4 = 4$ which gives $t_* = \sqrt{2} - 1$, and

$$x_* = x(t_*) = (\sqrt{2}-1, 9\sqrt{2}-12, 8-5\sqrt{2}, 8-5\sqrt{2})^T.$$

Finally, the equation (3.6) gives the stabilizing value $c = (c_1, c_2, c_3)^T = (6-5\sqrt{2}, 13-10\sqrt{2}, 1-\sqrt{2})^T$.

Example 2. Consider

$$G(s) = \frac{s-3}{s^2-4s-5}, \quad C(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2}.$$

The hyperplane $\mathcal{A}$ has the equation $x_1 + 3x_2 + 9x_3 + 27x_4 + 153 = 0$ and $\langle N, v^i \rangle + 153 > 0$ for all vertices v^i $(i = 1, \ldots, 5)$. By Corollary 1, there is no a stabilizing parameter c.

Remark 2. *In some control problems it is required that the stabilizing vector varies in some box, not in the whole space* $\mathbb{R}^l$. *In this case, the set* $\mathcal{A}$ *is not a hyperplane. In this case, the above result* (*Theorem* 1) *is not applicable and the problem remains open.*

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Taner Büyükköroğlu
Received September 31, 2016
Associate Professor Doctor
Department of Mathematics, Faculty of Science,
Anadolu University, 26470 Eskisehir, Turkey
e-mail: tbuyukkoroglu@anadolu.edu.tr

Gökhan Çelebi
Research Assistant Doctor
Department of Mathematics, Faculty of Arts and Sciences,
Bozok University, 66100 Yozgat, Turkey
e-mail: gokhan.celebi@bozok.edu.tr

Vakif Dzhafarov
Professor Doctor
Department of Mathematics, Faculty of Science,
Anadolu University, 26470 Eskisehir, Turkey
e-mail: vcaferov@anadolu.edu.tr