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ON THE REPRESENTATION OF UPPER SEMICONTINUOUS FUNCTIONS DEFINED ON INFINITE-DIMENSIONAL NORMED SPACES AS LOWER ENVELOPES OF FAMILIES OF CONVEX FUNCTIONS

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It is well known that a real-valued function defined on a metric space is upper (lower) semicontinuous if and only if it is a lower (upper) envelope of a family of continuous functions. In this paper, for functions defined on real normed spaces, this classical result is refined as follows. An upper (lower) bounded real-valued function defined on a normed space is upper (lower) semicontinuous if and only if it can be represented as a lower (upper) envelope of a family of convex (concave) functions that satisfy the Lipschitz condition on the whole space. It is shown that the requirement of upper (lower) boundedness may be omitted for positively homogeneous functions: a positively homogeneous function defined on a normed space is upper (lower) semicontinuous if and only if it is a lower (upper) envelope of a family of continuous sublinear (superlinear) functions. This characterization extends to arbitrary normed spaces a similar statement proved earlier by V. F. Demyanov and A. M. Rubinov for positively homogeneous functions defined on finite-dimensional spaces and later extended by A. Uderzo to the case of uniformly convex Banach spaces. The latter result allows to extend the notions of upper and lower exhausters introduced by V. F. Demyanov in finite-dimensional spaces to nonsmooth functions defined on arbitrary real normed spaces.

Keywords: semicontinuous functions, upper and lower envelopes, convex and concave functions, Lipschitz continuity, positively homogeneous functions.

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