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ON $S\Phi$-EMBEDDED SUBGROUPS OF FINITE GROUPS[1]

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A subgroup H of G is called $S\Phi$-embedded in G if there exists a normal subgroup T of G such that HT is S-quasinormal in G and $(H\cap T)H_G/H_G \leq \Phi(H/H_G)$, where H_G is the maximal normal subgroup of G contained in H and $\Phi(H/H_G)$ is the Frattini subgroup of H/H_G. In this paper, we investigate the influence of $S\Phi$-embedded subgroups on the structure of finite groups. In particular, some new characterizations of p-supersolvability of finite groups are obtained by assuming some subgroups are $S\Phi$-embedded.

Keywords: sylow p-subgroup; $S\Phi$-embedded subgroup; p-supersolvable group; p-nilpotent group.

Л. Чжан, В.Ġо, Л. Хо. О $S\Phi$-вложенных подгруппах конечных групп.

Подгруппа H группы G называется $S\Phi$-вложенной в G, если в G существует нормальная подгруппа T такая, что HT является S-квазинормальной в G и $(H\cap T)H_G/H_G \leq \Phi(H/H_G)$, где H_G — максимальная нормальная подгруппа группы G, содержащаяся в H, и $\Phi(H/H_G)$ — подгруппа Фраттини группы H/H_G. Изучается влияние $S\Phi$-вложенных подгрупп на структуру конечных групп. В частности, получены новые характеризации p-сверхразрешимости конечных групп в предположении, что некоторые подгруппы являются $S\Phi$-вложенными.

Ключевые слова: силовская p-подгруппа, $S\Phi$-вложенная подгруппа, p-сверхразрешимая группа, p-нильпотентная группа.

1. Introduction

Throughout this paper, all groups are finite and G denotes a finite group. All unexplained notation and terminology are standard, as in [8; 9] and [12].

It is well-known that embedded subgroups play an important role in the study of finite groups. Recall that a subgroup H of G is said to be quasinormal [5–7] (resp. S-quasinormal [19]) in G if H permutes with all subgroups (resp. Sylow subgroups) of G. A subgroup H of G is said to be *n-embedded* [10] in G if for some normal subgroup T of G and some S-quasinormal subgroup S of G contained in H, HT is normal in G and $H\cap T \leq S$. Let $\mathfrak{F}$ be a non-empty formation (see [8] or [9]). A subgroup H of G is said to be $\mathfrak{F}_s$*-quasinormal* [11] in G if for some normal subgroup T of G, HT is S-quasinormal in G and $(H\cap T)H_G/H_G \leq Z_{\mathfrak{F}}(G/H_G)$. A p-subgroup H of G is called *sn-embedded* [16] in G if for some normal subgroup T of G and some S-quasinormally embedded subgroup S of G contained in H, HT is S-quasinormal in G and $H\cap T \leq S$. Note that a subgroup H of G is said to be τ-quasinormal [15] in G if H permutes with all Sylow q-subgroups Q of G such that $(q, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$. A subgroup H of G is said to be *weakly τ-embedded* [4] in G if for some normal subgroup T of G and some τ-quasinormal subgroup S of G contained in H, HT is S-quasinormal in G and $H\cap T \leq S$. By using the above embedding subgroups, a series of interesting results were obtained (see [4; 10; 11; 16]).

Now we introduce the following new embedded subgroup.

D e f i n i t i o n. A subgroup H of G is said to be $S\Phi$-embedded in G if there exists a normal subgroup T of G such that HT is S-quasinormal in G and $(H\cap T)H_G/H_G \leq \Phi(H/H_G)$, where H_G is the maximal normal subgroup of G contained in H and $\Phi(H/H_G)$ is the Frattini subgroup of H/H_G.

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In this paper, we will use the $S\Phi$-embedded subgroup to study the structure of finite groups. Some new characterizations of p-supersolvability of finite groups are obtained.

2. Preliminaries

Lemma 2.1. *Let N be a normal subgroup of G and $H \leq G$.*

(1) *Assume that P is a non-trivial p-subgroup of G for some prime divisor p of $|G|$. Then $\Phi(P)N/N \leq \Phi(PN/N)$.*

(2) *Let H be a subgroup of G satisfying $(|H|, |N|) = 1$. Then $\Phi(H)N/N = \Phi(HN/N)$.*

(3) *If H is subnormal in G and H is a π-subgroup of G, then $H \leq O_\pi(G)$.*

P r o o f. (1) and (2) are clear. (3) is well known (see [9, 1.10.17)]).

Lemma 2.2. *Let G be a group and $H \leq K \leq G$.*

(1) *H is $S\Phi$-embedded in G if and only if G has a normal subgroup T such that HT is S-quasinormal in G, $H_G \leq T$ and $(H \cap T)/H_G \leq \Phi(H/H_G)$.*

(2) *Suppose that H is $S\Phi$-embedded in G, then H is $S\Phi$-embedded in K.*

(3) *Let H be a normal subgroup of G. Then K/H is $S\Phi$-embedded in G/H if and only if K is $S\Phi$-embedded in G.*

(4) *Suppose that H is normal in G, then for every $S\Phi$-embedded subgroup E of G satisfying $(|H|, |E|) = 1$, EH/H is $S\Phi$-embedded in G/H.*

P r o o f. (1) Assume that H is $S\Phi$-embedded in G and let T_0 be a normal subgroup of G such that HT_0 is S-quasinormal in G and $(H \cap T_0)H_G/H_G \leq \Phi(H/H_G)$. Let $T = T_0H_G$. Then $HT = HT_0$ is S-quasinormal in G and $(H \cap T)/H_G = (H \cap T_0)H_G/H_G \leq \Phi(H/H_G)$. The converse is clear.

(2) Suppose that T is a normal subgroup of G such that HT is S-quasinormal in G, $H_G \leq T$ and $(H \cap T)/H_G \leq \Phi(H/H_G)$. Let $T_1 = T \cap K$. Then $HT_1 = HT \cap K$ is S-quasinormal in K (see [1, 1.2.14(4)]). Since $H_G \leq H_K$, $(H \cap T_1)H_K/H_K = (H \cap T)H_K/H_K \leq \Phi(H/H_K)$ by [8, A, (9.2)(e)].

(3) Firstly, assume that K/H is $S\Phi$-embedded in G/H. By (1), G/H has a normal subgroup T/H such that $(K/H)(T/H)$ is S-quasinormal in G/H, $(K/H)_{G/H} = K_G/H \leq T/H$ and $(K/H \cap T/H)/(K/H)_{G/H} \leq \Phi((K/H)/(K/H)_{G/H})$. Then KT is S-quasinormal in G by [1, 1.2.14(1)]. Since $\Phi((K/H)/(K/H)_{G/H}) \cong \Phi(K/K_G)$, $(K \cap T)/K_G \leq \Phi(K/K_G)$. Hence K is $S\Phi$-embedded in G. Analogously, one can show that if K is $S\Phi$-embedded in G, then K/H is $S\Phi$-embedded in G/H.

(4) Assume that H is normal in G and E is $S\Phi$-embedded in G satisfying $(|H|, |E|) = 1$. By (1), G has a normal subgroup T such that ET is S-quasinormal in G, $E_G \leq T$ and $(E \cap T)/E_G \leq \Phi(E/E_G)$. By (3), we only need to show that HE is $S\Phi$-embedded in G. Clearly HET is S-quasinormal in G by [1, 1.2.2]. Since $(|H|, |E|) = 1$, $(|HE \cap T : H \cap T|, |HE \cap T : E \cap T|) = 1$. Hence $HE \cap T = (H \cap T)(E \cap T)$ by [8, A, (1.6)(b)], and $(HE \cap T)(HE)_G/(HE)_G \leq \Phi(HE/(HE)_G)$ by [8, A, (9.2)] and Lemma 2.1(2). Thus HE is $S\Phi$-embedded in G.

Lemma 2.3. *Assume that $G = N_1 \times N_2 \cdots \times N_t$, where $t \geqslant 1$ is an integer and $N_1, N_2, \cdots, N_t$ are non-abelian simple groups. Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $p \mid |N_i|$ for $i = 1, 2, \cdots, t$. Then any non-identity subgroup of $P \cap N_i$ can not be $S\Phi$-embedded in G, for $i = 1, 2, \cdots, t$.*

P r o o f. Assume that there exists an integer $i \in \{1, 2, \cdots, t\}$ such that $P \cap N_i$ has a non-trivial subgroup H which is $S\Phi$-embedded in G. Then H is $S\Phi$-embedded in N_i by Lemma 2.2(2). Obviously $H_{N_i} = 1$. Let T be a normal subgroup of N_i such that HT is S-quasinormal in N_i and $H \cap T \leq \Phi(H)$. If $T = 1$, then $H = HT$ is S-quasinormal in N_i. Hence $H \leq O_p(N_i) = 1$

by [1, 1.2.14(3)] and Lemma 2.1(3). This contradiction shows that $T = N_i$. It follows that $H = H \cap N_i \leq \Phi(H)$, a contradiction.

Lemma 2.4 [2, Theorem 7]. *Let G be a group whose Sylow p-subgroups are of order p^2 and $O_{p'}(G) = 1$. If G has the unique minimal normal subgroup N and N is isomorphic to a cyclic group of order p, then G is p-supersolvable.*

For a formation $\mathcal{F}$ of groups, $G^{\mathcal{F}}$ denotes the $\mathcal{F}$-residual of G, that is, $G^{\mathcal{F}} = \bigcap\{N|G/N \in \mathcal{F}\}$.

Lemma 2.5 [3, Theorem 1]. *Let $\mathcal{F}$ be a saturated formation and G be a group such that $G \notin \mathcal{F}$ but all its proper subgroups belong to $\mathcal{F}$. Then $G^{\mathcal{F}}\Phi(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$.*

The following facts about the generalized Fitting subgroup are useful in our proofs (see [13, Chapter X] and [10, Lemma 2.14, 2.15 and 2.16]).

Lemma 2.6. *Let G be a group. Then*:
(1) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
(2) *$C_G(F^*(G) \leq F(G)$.*
(3) *If N is a normal solvable subgroup of G, then $F^*(G/\Phi(N)) = F^*(G)/\Phi(N)$.*
(4) *If N is normal in G, then $F^*(N) = F^*(G) \cap N$.*
(5) *If P is a normal p-subgroup of G contained in $Z(G)$, then $F^*(G/P) = F^*(G)/P$.*

3. Main results

Theorem 3.1. *Let P be a Sylow p-subgroup of G for a fixed prime divisor p of $|G|$. Suppose that all maximal subgroups of P are $S\Phi$-embedded in G. Then G is p-supersolvable or $|P| = p$.*

P r o o f. Suppose that the assertion is false and let G be a counterexample of minimal order. Then $|P| \geqslant p^2$. We proceed the proof via the following steps:
(1) *G is not a non-abelian simple group* (It follows directly from Lemma 2.3).
(2) $O_{p'}(G) = 1$.
Suppose that $O_{p'}(G) > 1$. Then $G/O_{p'}(G)$ satisfies the hypothesis for G by Lemma 2.2(4). The choice of G implies that $G/O_{p'}(G)$ is p-supersolvable. It follows that G is p-supersolvable, a contradiction.
(3) *G is not p-solvable. Consequently $O_p(G) < P$.*
Assume that G is p-solvable. Then there exists a minimal normal subgroup N of G contained in $O_p(G)$ by (2). We claim that G/N is p-supersolvable. It is obvious if $|P/N| \leqslant p$. We may, therefore, assume that $|P/N| \geqslant p^2$. Then G/N satisfies the hypothesis by Lemma 2.2(3). Hence G/N is p-supersolvable. This implies that G has a unique minimal normal subgroup, N say, and $\Phi(G) = 1$. Then $G = N \rtimes M$ for some maximal subgroup M of G and $O_p(G) \cap M = 1$. Thus $O_p(G) = N(O_p(G) \cap M) = N$ and $P = O_p(G) \rtimes M_p$, where $M_p = P \cap M$. Let P_1 be a maximal subgroup of P containing M_p. Clearly $(P_1)_G = 1$. By the hypothesis, G has a normal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \leq \Phi(P_1)$. If $T = 1$, then $P_1 = P_1T$ is S-quasinormal in G. It follows from [1, 1.2.16] that P_1 is normal in G, a contradiction. Hence $O_p(G) \leq T$, so $P_1 = M_p(P_1 \cap O_p(G)) = M_p\Phi(P_1) = M_p$. Consequently $|O_p(G)| = p$. Therefore G is p-supersolvable, a contradiction.
(4) *If $O_p(G) > 1$, then $|P/O_p(G)| = p$ and $O_p(G)$ is a minimal normal subgroup of G.*
Assume that $|P/O_p(G)| \geqslant p^2$. Then $G/O_p(G)$ is p-supersolvable by Lemma 2.2(3) and the choice of G. It follows that G is p-solvable, a contradiction. Hence $|P/O_p(G)| = p$. Now assume that N is a non-trivial normal subgroup of G such that $N < O_p(G)$. Then $|P/N| \geqslant p^2$, and G/N is p-supersolvable similar as above, which also implies G is p-solvable, which contradicts (3).

(5) *If N is a minimal normal subgroup of G, then G/N is p-supersolvable or $|G/N|_p = p$.*

By (2), $p \mid |N|$. If N is abelian, then $N = O_p(G)$ and $|G/N|_p = p$ by (4). Now assume that N is non-abelian. Without loss of generality, we may assume $|PN/N| \geqslant p^2$.

Let M/N be an arbitrary maximal subgroup of PN/N. Then $M = NP_1$, where $P_1 = P \cap M$ is a maximal subgroup of P. We now show that M/N is $S\Phi$-embedded in G/N. By Lemma 2.2(3) we only need to show that M is $S\Phi$-embedded in G. By the hypothesis, G has a normal subgroup T such that P_1T is S-quasinormal in G, $(P_1)_G \leq T$ and $(P_1 \cap T)/(P_1)_G \leq \Phi(P_1/(P_1)_G)$. Obviously, MT is S-quasinormal in G by [1, 1.2.2]. If $N \leq T$, then $(M \cap T)M_G/M_G = (P_1 \cap T)M_G/M_G \leq \Phi(M/M_G)$ by Lemma 2.1(1) and [8, A, (9.2)]. Thus M is $S\Phi$-embedded in G. Hence we may assume that $N \cap T = 1$. If $P_1T \cap N = 1$, then $P_1T \cap N = (P_1 \cap N)(T \cap N) = 1$. Thus $P_1N \cap TN = (P_1 \cap T)N$ by [8, A, (1.2)]. Similar as above, M is $S\Phi$-embedded in G by taking the normal subgroup TN. Now assume that $P_1T \cap N \neq 1$. Since $|P_1T \cap N| = |P_1T \cap N : P_1T \cap N \cap T| = |P_1 \cap NT : P_1 \cap T|$ is power of p, $P_1T \cap N \leq O_p(G)$ by [1, 1.2.14(3)] and Lemma 2.1(3). It follows that $P_1 \cap N$ is subnormal in G. Hence $P_1 \cap N = N$ since $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. Consequently $N \leq P_1$, a contradiction. The above shows that G/N satisfies the hypothesis. Therefore (5) holds by the choice of G.

(6) *If N is a minimal normal subgroup of G, then $1 < P \cap N < P$.*

In view of (2), we see that $1 < P \cap N$. If $P \leq N$, then N is p-supersolvable since N satisfies the hypothesis by Lemma 2.2(2). But since G/N is a p'-group, G is p-solvable, which contradicts (3). Hence $P \cap N < P$.

(7) *G has a unique minimal normal subgroup, N say.*

By (1), G has a minimal normal subgroup. Now suppose that N_1 and N_2 are two different minimal normal subgroups of G. Then G/N_i is p-supersolvable or $|G/N_i|_p = p$ for $i = 1, 2$ by (5).

First assume that both G/N_1 and G/N_2 are p-supersolvable, then $G \cong G/(N_1 \cap N_2)$ is p-supersolvable, a contradiction. Secondly assume that G/N_1 is p-supersolvable and $|G/N_2|_p = p$. By (6), $p \mid |N_2|$. The G-isomorphism $N_2 \cong N_2N_1/N_1$ implies that $|N_2| = p$, so $|P| = p^2$. Therefore $N_1 \cap P$ is a maximal subgroup of P by (6) again, and N_1 is a non-abelian simple group since $N_1 \cap P$ is a Sylow p-subgroup of N_1. Hence $N_1 \cap P$ is $S\Phi$-embedded in N_1 by Lemma 2.2(2), which contradicts Lemma 2.3. Lastly suppose that $|G/N_1|_p = |G/N_2|_p = p$. Then $|P \cap N_1| = |P \cap N_2| = p$ and $|P| = p^2$. Consequently N_1 is a non-abelian simple group and $P \cap N_i$ is $S\Phi$-embedded in N_i for $i = 1, 2$ by Lemma 2.2(2), which contradicts Lemma 2.3 again. Hence we have (7).

(8) $O_p(G) = 1$.

Assume that $O_p(G) \neq 1$. Then by (4) and (7), $O_p(G)$ is a maximal subgroup of P and $O_p(G) = N$. If $O_p(G)$ is the unique maximal subgroup of P, then $O_p(G) = \Phi(P)$ and P is cyclic. Consequently $|O_p(G)| = p$ and $|P| = p^2$. Hence by Lemma 2.4, G is p-supersolvable, a contradiction. Thus P has a non-trivial maximal subgroup P_1 which is different from $O_p(G)$. Clearly $P = P_1O_p(G)$ and $(P_1)_G = 1$ by (7). Let T be a normal subgroup of G such that P_1T is S-quasinormal in G and $P_1 \cap T \leq \Phi(P_1)$. If $T = 1$, then P_1 is S-quasinormal in G, which implies $P_1 \trianglelefteq G$ by [1, 1.2.16], a contradiction. Also $T = G$ induces that $P_1 = P_1 \cap T \leq \Phi(P_1)$, a contradiction. Therefore $O_p(G) \leq T < G$, and so $P \leq P_1T$. If $P_1T < G$, then by [1, 1.2.14(3)], G has a proper normal subgroup L such that $P_1T \leq L$. Then L is p-supersolvable by the choice of G and Lemma 2.2(2). But then G is p-solvable since G/L is a p'-group, which contradicts (3). Thus $G = P_1T$. Clearly $O_p(G) \leq P \cap T$. If $P \cap T = P$, then $G = T$, a contradiction. Hence $O_p(G) = P \cap T$ is the Sylow p-subgroup of T. By the Schur-Zassenhaus Theorem, $T = O_p(G) \rtimes T_{p'}$, where $T_{p'}$ is a Hall p'-subgroup of T. Clearly $O_p(G) \nleq \Phi(G)$, so $\Phi(G) = 1$. Let M be a maximal subgroup of G such that $G = O_p(G) \rtimes M$. Then $P = O_p(G) \rtimes M_p$, where $M_p = P \cap M > 1$. Let P_2 be a maximal subgroup of P containing M_p. Then $(P_2)_G = 1$ and G has a normal subgroup K such that P_2K is S-quasinormal in G and $P_2 \cap K \leq \Phi(P_2)$. Similar as the above, we can obtain that $O_p(G) \leq K < G$. Thus $P_2 = (P_2 \cap O_p(G))M_p = \Phi(P_2)M_p = M_p$. This implies that $|O_p(G)| = p$, which contradicts Lemma 2.4. Hence we have (8).

(9) *Final contradiction.*

Let P_1 be a maximal subgroup of P containing $N_p = P \cap N$. Then $(P_1)_G = 1$ by (8). Hence by the hypothesis, G has a normal subgroup T such that P_1T is S-quasinormal in G and $P_1 \cap T \leq \Phi(P_1)$. If $T = 1$, then P_1 is normal in G by [1, 1.2.16], a contradiction. Hence $T \neq 1$ and so $N \leq T$. Consequently $P \cap N \leq P_1 \cap T \leq \Phi(P_1) \leq \Phi(P)$. This implies that N is p-nilpotent (see [12, IV, 4.7]), which contradicts (2) and (8). The final contradiction completes the proof.

Corollary 3.1.1. *Let p be a prime divisor of $|G|$ and H be a p-nilpotent normal subgroup of G such that G/H is p-supersolvable. If H has a Sylow p-subgroup P such that every maximal subgroup of P is $S\Phi$-embedded in G, then G is p-supersolvable.*

P r o o f. Suppose that the corollary is false and let (G, H) be a counterexample such that $|G| + |H|$ is minimal. Then clearly $p^2 \mid |H|$.

Firstly we claim that $H = P$. Since $O_{p'}(H)$ is normal in G, $(G/O_{p'}(H), H/O_{p'}(H))$ satisfies the hypothesis by Lemma 2.2(4). The choice of (G, H) implies that $G/O_{p'}(H)$ is p-supersolvable if $O_{p'}(H) > 1$. It follows that G is p-supersolvable, a contradiction. Thus $H = P$.

Now we prove that H is a minimal normal subgroup of G. Let N be a minimal normal subgroup of G contained in H. We first show that G/N is p-supersolvable. If $|H/N| \leqslant p$, this is clear. Hence we may assume that $|H : N| \geqslant p^2$. Then G/N is also p-supersolvable by considering $(G/N, H/N)$ and using Lemma 2.2(3). This implies that G has a unique minimal normal subgroup N contained in H and $N \nsubseteq \Phi(G)$. Let M be a maximal subgroup of G such that $G = N \rtimes M$. Since $H \cap M$ is normal in G (see [8, A, (8.4)]), $H \cap M = 1$. Thus $H = N(H \cap M) = N$.

Let H_1 be a non-trivial maximal subgroup of H such that $H_1 \trianglelefteq G_p$ for some Sylow p-subgroup G_p of G. By the hypothesis, G has a normal subgroup T such that H_1T is S-quasinormal in G and $H_1 \cap T \leq \Phi(H_1)$. If $H \cap T = 1$, then $H_1 = H_1(H \cap T) = H \cap H_1T$ is S-quasinormal in G (see [1, 1.2.19]). Hence H_1 is normal in G by [1, 1.2.16], a contradiction. Thus $H \cap T \neq 1$, and so $H \leq T$. This implies that $H_1 = H_1 \cap T \leq \Phi(H_1)$ and therefore $H_1 = 1$. Then $|H| = p$ and so G is p-supersolvable. The contradiction completes the proof.

Corollary 3.1.2. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is $S\Phi$-embedded in G, then G is p-nilpotent.*

P r o o f. It follows directly from Theorem 3.1 and [18, (10.1.8)].

Corollary 3.1.3. *Let E be a normal subgroup of G such that G/E is p-nilpotent, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If E has a Sylow p-subgroup P such that every maximal subgroup of P is $S\Phi$-embedded in G, then G is p-nilpotent.*

P r o o f. First assume that $E = P$ and let K/P be the normal Hall p'-subgroup of G/P. Then $K = P \rtimes K_{p'}$ by the Schur-Zassenhaus Theorem, where $K_{p'}$ is a Hall p'-subgroup of K. Clearly $K_{p'}$ is also a Hall p'-subgroup of G. By Lemma 2.2(2) and Corollary 3.1.2, K is p-nilpotent, and so $K = P \times K_{p'}$. It follows that $K_{p'}$ is normal in G, and consequently G is p-nilpotent. Now assume that $E > P$. Then E is p-nilpotent by Lemma 2.2(2) and Corollary 3.1.2. Let $E_{p'}$ be the normal Hall p'-subgroup of E. By Lemma 2.2(4) and induction, $G/E_{p'}$ is p-nilpotent. This also implies that G is p-nilpotent.

Corollary 3.1.4. *Suppose that all maximal subgroups of every non-cyclic Sylow subgroup of G are $S\Phi$-embedded in G. Then G is a Sylow tower group of supersolvable type.*

P r o o f. Let p be the smallest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If P is cyclic, then G is p-nilpotent by [18, 10.1.9]. Otherwise, G is still p-nilpotent by Corollary 3.1.2. Let U be the normal Hall p'-subgroup of G. By Lemma 2.2(2), U satisfies the hypothesis. Therefore, by induction, G is a Sylow tower group of supersolvable type.

Theorem 3.2. *Assume that all maximal subgroups of every non-cyclic Sylow subgroup of $F^*(G)$ are $S\Phi$-embedded in G. Then G is supersolvable.*

P r o o f. Suppose that the theorem is false and let G be a counterexample of minimal order. We proceed via the following steps:

(1) *Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.*

Let M be a proper normal subgroup of G containing $F^*(G)$. Then $F^*(M) = F^*(G)$ by Lemma 2.6(4). Hence M satisfies the hypothesis by Lemma 2.2(2). The choice of G implies that M is supersolvable.

(2) *G is not solvable.*

Assume that G is solvable, then $F^*(G) = F(G)$ by Lemma 2.6(1).

If $\Phi(G) = 1$, then $F(G) = N_1 \times N_2 \cdots \times N_t$ by [8, A, (10.6)], where $t \geqslant 1$ is an integer and $N_1, N_2, \cdots, N_t$ are minimal normal subgroups of G. Without loss of generality, assume that $P = N_1 \times N_2 \cdots \times N_s$ $(1 \leqslant s \leqslant t)$ is the Sylow p-subgroup of $F(G)$ for some prime $p \mid |F(G)|$. We claim that $|N_i| = p$ for $i = 1, 2, \cdots, s$. Otherwise, $|N_i| > p$ for some $i \in \{1, 2, \cdots, s\}$. Without loss of generality, assume that $|N_1| > p$. Let N_1^* be a maximal subgroup of N_1 such that $N_1^* \trianglelefteq G_p$ for some Sylow p-subgroup G_p of G. Let $P_1 = N_1^* N_2 \cdots N_s$. Then P_1 is a maximal subgroup of P and P_1 is normal in G_p with $(P_1)_G = N_2 \cdots N_s$. Put $D = (P_1)_G$. Since P_1 is $S\Phi$-embedded in G, G has a normal subgroup T such that P_1T is S-quasinormal in G, $D \leq T$ and $(P_1 \cap T)/D \leq \Phi(P_1/D)$. Since $\Phi(N_1^*) \leq \Phi(N_1) \leq \Phi(G) = 1$ and the G-isomorphism $P_1/D \cong N_1^*$, we have that $P_1 \cap T = D$. If $N_1 \leq T$, then $P \leq T$ and $P_1 = P_1 \cap T = D$, which implies $N_1^* = 1$, a contradiction. Hence $N_1 \cap T = 1$. Consequently $P \cap T = D$ and $P_1 = P_1(P \cap T) = P \cap P_1T$ is S-quasinormal in G (see [1, 1.2.19]), which implies that P_1 is normal in G by [1, 1.2.16], a contradiction. Therefore $F(G) = N_1 \times N_2 \cdots \times N_t$, where N_i $(i = 1, 2, \cdots, t)$ are all of prime order. But then $G/C_G(N_i)$ $(i = 1, 2, \cdots, t)$ is abelian, so $G/(\bigcap_{i=1}^t C_G(N_i)) = G/C_G(F(G))$ is abelian. Also $C_G(F(G)) = F(G)$ by Lemma 2.6(2). This implies that G is supersolvable since every chief factor of G below $F(G)$ is cyclic.

Now assume $\Phi(G) > 1$. Let $P = O_p(\Phi(G)) > 1$. Since $F^*(G/P) = F(G/P) = F(G)/P$ (see [8, A, (9.3)(c)]), G/P satisfies the assumptions by Lemma 2.2 (3), (4). The choice of G implies that G/P is supersolvable. Thus G is supersolvable, a contradiction.

(3) *$F^*(G) = F(G)$ and $G = F(G)O^p(G)$.*

By Lemma 2.2(2) and Corollary 3.1.4, $F^*(G)$ is a Sylow tower group of supersolvable type. Particularly, $F^*(G)$ is solvable. Hence $F^*(G) = F(G)$ by Lemma 2.6(1). Suppose that $F(G)O^p(G) < G$. Then $F(G)O^p(G)$ is supersolvable by (1). Thus G is solvable since $G/F(G)O^p(G)$ is a p-group, which contradicts (2). Hence (3) holds.

(4) *$\Phi(F(G)) = 1$ and $C_G(F(G)) = F(G)$.*

Assume that $F(G)$ has a Sylow p-subgroup P such that $\Phi(P) > 1$. By Lemma 2.6(3), $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$. So $G/\Phi(P)$ satisfies the hypothesis. The choice of G implies that $G/\Phi(P)$ is supersolvable. Consequently G is supersolvable. This contradiction shows that $F(G)$ is elementary abelian and so $\Phi(F(G)) = 1$. Then together with (3) and Lemma 2.6(2), we have $C_G(F(G)) = F(G)$.

(5) *There exists no normal subgroup of G contained in $F(G)$ with prime order.*

Assume that G has a normal subgroup N contained in $F(G)$ with $|N| = p$. Let $C = C_G(N)$. By (4), $F(G) \leq C$. If $C < G$, then C is supersolvable by (1). But since G/C is abelian, it follows that G is solvable, which contradicts (2). We may, therefore, assume $C = G$, that is, $N \leq Z(G)$. By Lemma 2.6(5), $F^*(G/N) = F^*(G)/N$. Hence G/N is supersolvable by Lemma 2.2 and the choice of G. It follows that G is supersolvable, a contradiction.

(6) $\pi(\Phi(G)) = \pi(F(G))$.

Suppose that (6) is false. Then $F(G)$ has a Sylow subgroup P such that $P \cap \Phi(G) = 1$. Similar as the proof in (2), we see that there exists at least one minimal normal subgroup of G contained in P with prime order, which contradicts (5).

(7) *$F(G)$ is a p-group and there exists exactly one minimal normal subgroup of G contained in $F(G)$, L say.*

Suppose that $|F(G)|$ contains two different primes p and q. Let P and Q be the Sylow p-subgroup and the Sylow q-subgroup of $F(G)$. By (6), G has a minimal normal subgroup L contained in $P \cap \Phi(G)$. By [13, p.128], $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Denote $E(G/L) = E/L$. Since $F(G/L) = F(G)/L$ by [8, A, (9.3)(c)], $[Q, E] \leq Q \cap L = 1$. It follows from (4) that $F(G)E \leq C_G(Q)$. If $C_G(Q) < G$, then $C_G(Q)$ is supersolvable by (1). Hence $F^*(G/L) = F(G)/L$ by Lemma 2.6(1). The choice of G and Lemma 2.2 imply that G/L is supersolvable. Consequently G is supersolvable, a contradiction. Therefore $C_G(Q) = G$, which contradicts (5). Thus $F(G)$ is a p-group.

Now assume that X is another minimal normal subgroup of G contained in $F(G)$ different from L. Using the same symbol as above, then $[X, E] \leq X \cap L = 1$, and so $F(G)E \leq C_G(X)$. If $C_G(X) < G$, then $C_G(X)$ is supersolvable by (1). Similar as above, we see that G is supersolvable, a contradiction. Therefore $C_G(X) = G$, which contradicts (5). Thus L is the unique minimal normal subgroup of G contained in $F(G)$.

(8) *Final contradiction.*

By (4), there exists a maximal subgroup P_1 of $P = F(G)$ which does not contain L and $\Phi(P_1) = 1$. Then $(P_1)_G = 1$ by (7). Let T be a normal subgroup of G such that P_1T is S-quasinormal in G and $P_1 \cap T = 1$. Since $|P \cap T| = |P \cap T : P_1 \cap T| \leqslant |P : P_1| = p$, $P \cap T = 1$ by (5). Hence $P_1 = P_1(P \cap T) = P \cap P_1T$ is S-quasinormal in G by [1, 1.2.19]. Thus P_1 is normal in G by (3) and [1, 1.2.16]. The final contradiction completes the proof.

Corollary 3.2.1 [17, Theorem 3.1]. *Let G be a solvable group. If all maximal subgroups of the Sylow subgroups of $F(G)$ are normal in G, then G is supersolvable.*

Note that a subgroup H of G is called *c-normal* [20] in G if there exists a normal subgroup N of G such that $HN = G$ and $H \cap N \leq H_G$. Clearly a c-normal subgroup H is $S\Phi$-embedded in G. But the following example shows that the converse is false.

Example 3.1. *Let $G = S_4$ and $H = \langle(1234)\rangle$. Clearly $H_G = 1$ and $\Phi(H) = \{1, (13)(24)\}$. It is easy to check that H is $S\Phi$-embedded in G by taking the Klein 4-group K_4. However H is not c-normal in G since G has no normal subgroup of order 6.*

Corollary 3.2.2 [14, Theorem 1]. *Assume that G is solvable and every maximal subgroup of Sylow subgroups of $F(G)$ is c-normal in G. Then G is supersolvable.*

Theorem 3.3. *Let E be a normal subgroup of G such that G/E is p-supersolvable, where p is a prime divisor of $|G|$. If every cyclic subgroup of E with order p or 4 (if $p = 2$) is $S\Phi$-embedded in G, then G is p-supersolvable.*

P r o o f. Suppose that the theorem is false and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. Note that $\mathcal{U}^p$ denotes the class of p-supersolvable groups. Then:

(1) *$p \mid |E|$ and $E = G^{\mathcal{U}^p}$* (It follows directly from the choice of (G, E)).

(2) *G is a minimal non-p-supersolvable group and $O_{p'}(G) = 1$.*

It follows from Lemma 2.2 (2), (4) and the choice of (G, E).

(3) *$G/\Phi(G)$ is a non-abelian simple group.*

By (2) and Lemma 2.5, $G^{\mathcal{U}^p}\Phi(G)/\Phi(G)$ is the unique minimal normal subgroup of $G/\Phi(G)$. Let $N = G^{\mathcal{U}^p}\Phi(G)$. Then G/N is p-supersolvable. Hence $p \mid |N/\Phi(G)|$.

Assume that $N/\Phi(G)$ is abelian. Then N is solvable. It follows from (2) and [9, (3.4.2)] that $G^{\mathcal{U}^p}$ is a p-group and $G^{\mathcal{U}^p}/\Phi(G^{\mathcal{U}^p})$ is a non-cyclic G-chief factor with exponent p or 4 (if $G^{\mathcal{U}^p}/\Phi(G^{\mathcal{U}^p})$

is a non-abelian 2-group). Take $x \in G^{\mathcal{U}^p} \setminus \Phi(G^{\mathcal{U}^p})$ such that $\langle x\rangle\Phi(G^{\mathcal{U}^p})$ is normal in some Sylow p-subgroup of G. Denote $H = \langle x\rangle$. Then H is of order p or 4. If H is normal in G, then $G^{\mathcal{U}^p}/\Phi(G^{\mathcal{U}^p}) = H\Phi(G^{\mathcal{U}^p})/\Phi(G^{\mathcal{U}^p})$ is cyclic of order p, a contradiction. Thus $H_G = 1$ or $H_G = \langle x^2\rangle = \Phi(H)$ if $|H| = 4$. By the hypothesis, G has a normal subgroup T such that HT is S-quasinormal in G, $H_G \le T$ and $(H \cap T)/H_G \le \Phi(H/H_G)$. Obviously $H \cap T \le \Phi(H)$ whether $H_G = 1$ or $H_G = \langle x^2\rangle$. Since $G^{\mathcal{U}^p}/\Phi(G^{\mathcal{U}^p})$ is a chief factor of G, $(T \cap G^{\mathcal{U}^p})\Phi(G^{\mathcal{U}^p}) = G^{\mathcal{U}^p}$ or $\Phi(G^{\mathcal{U}^p})$. In the case when $(T \cap G^{\mathcal{U}^p})\Phi(G^{\mathcal{U}^p}) = G^{\mathcal{U}^p}$, we have that $H = H \cap T \le \Phi(H)$, a contradiction. Hence $T \cap G^{\mathcal{U}^p} \le \Phi(G^{\mathcal{U}^p})$, so $H\Phi(G^{\mathcal{U}^p}) = (HT \cap G^{\mathcal{U}^p})\Phi(G^{\mathcal{U}^p})$ is S-quasinormal in G by [1, 1.2.19]. But then $H\Phi(G^{\mathcal{U}^p})$ is normal is G (see [1, 1.2.16]). Consequently $H\Phi(G^{\mathcal{U}^p}) = G^{\mathcal{U}^p}$ and so $G^{\mathcal{U}^p}/\Phi(G^{\mathcal{U}^p})$ is cyclic. This contradiction shows that $N/\Phi(G)$ is non-abelian. It follows from (2) that $G/\Phi(G) = N/\Phi(G)$ is a non-abelian simple group.

(4) *$F(G) = \Phi(G) = O_p(G) = Z(G)$.*

By (2) and (3), $F(G) = \Phi(G) = O_p(G) \ge Z(G)$. If $C = C_G(O_p(G)) < G$, then $C \le \Phi(G)$ by (3). Let M be an arbitrary maximal subgroup of G. Then $O_p(G) = \Phi(G) \le M = Z_{\mathcal{U}^p}(M)$ by (2), and so $O_p(G) \le Z_{\mathcal{U}}(M)$. Hence $M/C_M(O_p(G)) = M/(M \cap C) = M/C$ is supersolvable by [12, VI, 9.8]. This shows that G/C is a minimal non-supersolvable group. Then G/C is solvable (see [18, (10.3.4)]) and so G is solvable, which contradicts (3). Therefore $C = G$.

(5) *Final contradiction.*

Note that if every element of order p or 4 belongs to $\Phi(G) = Z(G)$, then G is p-nilpotent by [12, IV, 5.5], and so G is p-supersolvable. Hence there exists an element x in G of order p or 4, which does not belong to $\Phi(G)$. Let $H = \langle x\rangle$. If H is normal in G, then $H\Phi(G) = G$ by (3), and so $G = \langle x\rangle$ is cyclic, which is impossible. Then $H_G = 1$ or $H_G = \langle x^2\rangle = \Phi(H)$ (when $|H| = 4$). Let T be a normal subgroup of G such that HT is S-quasinormal in G, $H_G \le T$ and $(H \cap T)/H_G \le \Phi(H/H_G)$. Similar as above, we have that $H \cap T \le \Phi(H)$. By (3), $T = G$ or $T \le \Phi(G)$. If $T = G$, then $H = H \cap T \le \Phi(H)$, a contradiction. Hence $T \le \Phi(G)$. It follows from [1, 1.2.7(2) and 1.2.14(3)] that $HT\Phi(G)/\Phi(G) = H\Phi(G)/\Phi(G)$ is subnormal in $G/\Phi(G)$. Then $H\Phi(G) = G$ by (3) and the choice of x. Thus $G = \langle x\rangle$ is cyclic. This contradiction completes the proof.

СПИСОК ЛИТЕРАТУРЫ

1. **Ballester-Bolinches A., Esteban-Romero R., Asaad M.** Products of finite groups. Berlin; New York: Walter de Gruyter, 2010. 334 p.
2. **Ballester-Bolinches A., Ezquerro L. M., Skiba A. N.** On second maximal subgroups of Sylow subgroups of finite groups // J. Pure Appl. Algebra. 2011. Vol. 215, no. 4. P. 705–714.
3. **Ballester-Bolinches A., Pedraza-Aguilera M. C.** On minimal subgroups of finite groups // Acta Math. Hungar. 1996. Vol. 73, no. 4. P. 335–342.
4. **Chen X., Guo W.** On weakly S-embedded and weakly τ-embedded subgroups // Sib. Math. J. 2013. Vol. 54, no. 5. P. 931–945.
5. **Cossey J., Stonehewer S.** Abelian quasinormal subgroups of finite p-groups // J. Algebra. 2011. Vol. 326, no. 1. P. 113–121.
6. **Cossey J., Stonehewer S.** On the rarity of quasinormal subgroups // Rend. Semin. Mat. Univ. Padova. 2011. Vol. 125. P. 81–105.
7. **Deskins W. E.** On quasinormal subgroups of finite groups // Math. Z. 1963. Vol. 82. P. 125–132.
8. **Doerk K., Hawkes T.** Finite soluble groups. Berlin: Walter de Gruyter, 1992. 891 p.
9. **Guo W.** The theory of classes of groups. Beijing; New York; Dordrecht; Boston; London: Science Press-Kluwer Acad. Publ., 2000. 258 p.
10. **Guo W., Skiba A. N.** Finite groups with given s-embedded and n-embedded subgroups // J. Algebra. 2009. Vol. 321, no. 10. P. 2843–2860.
11. **Huang J.** On $\mathcal{F}_s$-quasinormal subgroups of finite groups // Comm. Algebra. 2010. Vol. 38, no. 11. P. 4063–4076.
12. **Huppert B.** Endliche Gruppen I. Berlin; New York: Springer-Verlag, 1967. 793 p.
13. **Huppert B., Blackburn N.** Finite groups III. Berlin; New York: Springer-Verlag, 1982. 454 p.

14. **Li D., Guo X.** The influence of c-normality of subgroups on the structure of finite groups II // Comm. Algebra. 1998. Vol. 26, no. 6. P. 1913–1922.
15. **Lukyanenko V. O., Skiba A. N.** On τ-quasinormal and weakly τ-quasinormal subgroups of finite groups // Math. Sci. Res. J. 2008. Vol. 12, no. 7. P. 243–257.
16. **Malinowska I. A.** Finite groups with sn-embedded or s-embedded subgroups // Acta Math. Hungar. 2012. Vol. 136, no.1–2. P. 76–89.
17. **Ramadan M.** Influence of normality on maximal subgroups of Sylow subgroups of a finite group // Acta Math. Hungar. 1992. Vol. 59, no. 1–2. P. 107–110.
18. **Robinson D. J. S.** A course in the theory of groups. Berlin; New York: Springer-Verlag, 1982. 499 p.
19. **Schmid P.** Subgroup permutable with all Sylow subgroups // J. Algebra. 1998. Vol. 207, no. 1. P. 285–293.
20. **Wang Y.** c-Normality of groups and its properties // J. Algebra. 1996. Vol. 180, no. 3. P. 954–965.

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